On the backreaction of quantum scalar fields on the de Sitter spacetime

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Master-Thesis
Faculty of Physics
Ludwig–Maximilians–University
Munich

written by
Lorenz Gärtner
from Vienna

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Supervisor: Prof. Dr. Ivo Sachs

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Abstract

In this work we analyse the behaviour of a quantised scalar field in a de Sitter background and the backreaction on the geometry. Due to the dynamic nature of the background no unique vacuum, diagonalising the Hamiltonian at all times, exists on the basis of which we can build a global Fock space. Imposing de Sitter invariance of our vacuum we find a two parameter class of vacua known in the literature as Mottola-Allen vacua, which are invariant under the time-preserving part of the de Sitter symmetry group. Furthermore, fixing one of the mentioned parameters results in a one parameter group of vacua, leaving argument-symmetric Green functions invariant under the full de Sitter group. Lastly, matching our result to the flat Minkowski solutions on very small scales where curvature is expected to be negligible, we can fix the last parameter and obtain what is referred to as the Bunch-Davies (BD) vacuum. We use our obtained insight to compute the expectation value of the regularised energy momentum tensor for a general choice of de Sitter invariant vacua. We conclude that as long as we respect de Sitter invariance, we do not get any dynamic backreaction and only obtain a constant shift in the cosmological constant.

We then look for the breaking of this isometry in the loop corrections of a self interacting $\lambda\phi^4$ theory, but find that it is generally also respected in loops, as long we restrict to a certain coordinate patch of de Sitter spacetime.

Finally, we break de Sitter isometry explicitly by introducing a scalar metric perturbation to the de Sitter geometry. We introduce a free massive scalar field to our spacetime and estimate the backreaction by solving for the perturbation. Our results show that in the short wavelength (UV) regime the largest contribution to the metric perturbation decays while oscillating in a similar way to gravitational wave modes. In the long wavelength (IR) regime we find that one part of the solution significantly grows and these perturbations can no longer be considered small.

Contents

Chapter 1

Introduction

Since the birth of general relativity, the question about what exact type of spacetime geometry we live in has relentlessly puzzled cosmologists. The uprise of the inflationary paradigm has put the de Sitter geometry in bright spotlight. It does a very good job at approximating an exponentially expanding phase of our universe with the additional bonus of being maximally symmetric. Exact application of this geometry although has been ruled out. One reason is the constant Hubble parameter, or exact Hubble law, which contradicts observation. Furthermore, we know that different forms of energy or matter contributions exist, which had their respective periods of domination in the past [1]. We are pretty certain that either some form of positive energy contribution must have existed already during the stage of inflation or that the inflaton field, which drives the expansion, decays into the standard model particles after inflation has ended. So de Sitter spacetime is only a good approximation as long as we can completely ignore any energy contribution not driving the expansion. Inflation acts as a classical smoothing mechanism, but can not directly smooth out quantum perturbations.
Furthermore, we know that the quantum nature of matter has lead to valuable results in terms of structure formation in the early universe by considering quantised perturbations on top of flat Minkowski spacetime [2]. In the past centuries, much work has been done in the intersection of gravity and quantum field theory. Up to this day, we have not found a way to describe gravity above the Planck scale. The usual approach is to investigate the quantised fields on a classical background other than flat Minkowski spacetime. This approach of coupling quantum fields to classical gravity has become a huge topic, which is most commonly known as the *semi-classical* approach. Here, the quantum contributions enter in the Einstein equation through quantum expectation values.
But why would one even expect microscopic quantum behaviour to have a significant impact on macroscopic scales? Generally, we expect that quantum effects should be highly suppressed on scales of order of the Hubble radius H^{-1}. In a static or non-expanding spacetime this generally holds true. But since de Sitter space is exponentially expanding, scales of order of the Planck length can become macroscopic over time. Hence, we can expect microscopic quantum behaviour to have a significant influence on large scales after some particular time [3]. We can easily estimate the time at which the Planck length grows to the size of the Hubble radius. Consider the scale factor of an exponentially expanding universe,

$$a(t) = a_0 \, e^{Ht}, \tag{1.1}$$

where a_0 is some initial length scale. Assuming that general relativity approximately remains valid to just below the Planck scale, the time for the Planck length, $l_p = \sqrt{G\hbar/c^3} \simeq 1.62 \times$

10^{-33} cm, to become of order of the Hubble radius is

$$t \sim H^{-1} \ln\left(\frac{H}{l_p}\right). \tag{1.2}$$

For a Hubble parameter of $H = 100\,km\,s^{-1}\,Mpc^{-1}$, $\ln\left(\frac{H}{l_p}\right) \sim 40$. Then the time when the Planck length becomes of order of the Hubble radius is around $t \sim 10^{19}s \sim 10^{11}yrs$. Although we do not require the Planck length to be blown up to quite so large scales for quantum effects to become significant macroscopically, we see that it is indeed possible.
In this text, we want to investigate explicitly the quantum behaviour of a real scalar field with respect to a de Sitter background. Due to the explicit time dependence of the metric, any vacuum choice will evolve during the evolution of the spacetime and therefore lose the property of being the lowest energy state. Particles defined with respect to the chosen vacuum will appear sourced by the dynamic motion of the background. In such cases, the definition of a vacuum is ambiguous and no preference can be given to one particular choice over any other. Although we can impose physically motivated assumptions such as invariance under the de Sitter group, which is the compliment of Poincaré invariance in flat space [4], or to match our theory to the flat space case, where we expect curvature effects to be negligible.
A few years back, a very interesting question has been raised by Krotov and Polyakov [5]: *What effect do the particles, created during the expansion of our universe, have on the cosmological constant?* From classical FLRW cosmology, it is clear that once we add any positive energy content to our universe, the gravitational pull slows the expansion and can even lead to a big crunch, depending on its relative magnitude compared to cosmological constant components. *So, can we expect that the particles created during the dynamic evolution of de Sitter spacetime backreact and slow down the expansion, therefore leading to a time dependent screening of the cosmological constant?*
Much work has been done to arrive at an answer to these questions [5–9]. But as already noted in the original paper [5] there are certain issues with their formulation. Due to the ambiguity of the vacuum state, the particle concept is ill-defined and the particle density at a given moment depends on the choice of vacuum. Hence, if we find the above mentioned effects, how can we be sure about their validity? Must we require to find vacuum independent solutions to believe our results? Another point mentioned in [5] is, how strong can these effects be, considering that particle densities get diluted during the exponential expansion of space? Even if we see such effects, can we expect them to be negligible?
In the formalism of semi-classical gravity we calculate the regularised quantum expectation value of the energy-momentum tensor (EMT) for a free scalar field in de Sitter spacetime. We find that when respecting de Sitter invariance, the contribution to the expectation value of the EMT will be constant and proportional to the metric and therefore lead to a constant shift in the cosmological constant. We find that to get time dependent contributions to the expectation value of the EMT, de Sitter invariance needs to be broken. Introducing scalar perturbations to the de Sitter metric, we calculate the backreaction on the spacetime. Our results show that for short wavelength modes the perturbations decay while exhibiting oscillatory behaviour. On the other hand the long wavelength modes include a growing solution which will eventually break perturbation theory.
To get a feeling for our background spacetime, we start by introducing the basics of de Sitter geometry in sec. 2 [1, 4, 8, 10–13]. Here the details of our geometry are discussed, and the reasons why physics on a dynamic background brings some complications with it, such as

non-conservation of energy. Moreover, we discuss the quantisation of a free scalar field on a curved background in sec. 3 [1, 4, 7, 8, 14–21]. Having discussed the general features, we specify to FLRW geometries out of cosmological motivation and look at some aspects of our theory, before we restrict ourselves to the de Sitter geometry. Due to the above mentioned non-conservation of energy and the ambiguity of the vacuum choice in a dynamic background, much of the discussion in this section is devoted to the different vacuum choices and the physical motivation behind them.

Having laid out the foundation of our understanding we turn to the interesting questions mentioned above, how and if quantum effects can backreact significantly on our spacetime geometry. We calculate the quantum expectation value of the EMT, which we expect to contribute to the Einstein equation, in sec. 4 [1, 10, 14, 15, 22–28]. We further investigate the nature of the contribution and what this means physically for different choices of vacua. Out of regularisation purposes we have to extend beyond conventional general relativity. What this means for inflation is discussed in sec. 5 [1, 10, 15, 22, 29, 30].

Having discussed the free theory and UV effects extensively, we turn to the other end of the energy spectrum with the same question in mind. In sec. 6 [1, 4, 5, 8, 9, 15, 23, 31–38] we discuss the IR limit of the previously obtained results and of loop corrections to the propagator. We investigate if a deviation from de Sitter invariance can occur due to such loop corrections. To this end we introduce the closed time path (CTP) formalism, which reduces the ambiguity of a vacuum choice slightly, as it allows us to do calculations by making reference to only one vacuum at one specific point in time.

Our results lead to the conclusion, that de Sitter invariance is respected also in loop effects in the Poincaré patch. Therefore we continue by breaking de Sitter invariance explicitly, by allowing for backreaction in the metric on a perturbative level. Considering scalar perturbations to the de Sitter background, we investigate their behaviour in the UV and IR limits in sec. 7.

Chapter 2

The de Sitter geometry

In this section we want to set the scene by introducing the most important features of de Sitter spacetime. Firstly, we give a physically motivated introduction in sec. 2.1 [1]. We continue to the more mathematical side and give the geometric definition of the de Sitter geometry in terms of a higher dimensional embedding space in sec. 2.2 [10, 11]. As we start to get a feeling for the geometry, we discuss symmetries in sec. 2.3 [4, 8], which play a very important role physically. In sec. 2.4 [8, 11, 12] we then describe the most useful coordinate patches. Finally in sec. 2.5 [12, 13] we discuss the complications arising from the lack of a timelike Killing vector in de Sitter spacetime for the vacuum choice and when quantising fields on a dynamic background in general.

2.1 A cosmological motivation for de Sitter

If one wants to build models of the universe as we observe it today, certain restrictions are posed upon the class of spacetime geometries that one can use. From a geometric viewpoint, the most important cosmological features of space are homogeneity and isotropy on large scales, known as the *cosmological principle* [1]. The spaces which reflect these properties fall into three categories: flat space, a spatial sphere of constant positive curvature and a spatial hyperboloid of constant negative curvature [1]. These cases are generally summarised in terms of the Friedmann-Lemaître-Robertson-Walker (FLRW) spacetime, where the invariant line element is usually written as

$$ds^2 = dt^2 - a^2(t)\left(\frac{dr^2}{1-Kr^2} + r^2 d\Omega_{d-2}^2\right) = g_{\mu\nu}dx^\mu dx^\nu, \tag{2.1}$$

where $d\Omega_{d-2}^2$ is the line element of a $d-2$ dimensional sphere and $a(t)$ is the scale factor, which gives the Hubble parameter, $H = \frac{\dot{a}}{a} = \frac{1}{a}\frac{\mathrm{d}\,a}{\mathrm{d}t}$. Here $K = 1$ describes a spherical space of constant positive curvature, $K = -1$ the hyperbolic space of constant negative curvature and $K = 0$ the flat case.

It is very useful to define the conformal time by

$$dt = a\,d\eta, \tag{2.2}$$

so that the we can write the line element as

$$ds^2 = a^2(\eta)\left(d\eta^2 - \frac{dr^2}{1-Kr^2} + r^2 d\Omega_{d-2}^2\right). \tag{2.3}$$

This is useful as for $K = 0$ we are just working in a conformally flat spacetime.
De Sitter spacetime is the FLRW solution of a universe dominated by the cosmological constant. In this case the Einstein equation becomes

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R - \Lambda g_{\alpha\beta} = 0. \tag{2.4}$$

Upon contraction we can easily see, that de Sitter spacetime has constant positive curvature[1] (for $d > 2$) and $\Lambda > 0$,

$$R = -\frac{2d}{d-2}\Lambda. \tag{2.5}$$

Substituting back into the Einstein equation, we see that the Ricci tensor is proportional to the metric,

$$R_{\alpha\beta} = -\frac{2}{d-2}\Lambda g_{\alpha\beta}. \tag{2.6}$$

On cosmological scales, the EMT of a perfect fluid, specified by its energy density ρ, pressure p and d-velocity u^μ,

$$T^{\mu\nu} = (\rho + p)u^\mu u^\nu - g^{\mu\nu} p, \tag{2.7}$$

is usually a good approximation to describe matter in our universe. From this EMT we see, that we could realize the cosmological constant also by the equation of state $\rho = -p = const.$, driving the expansion of the universe. Solving the different components of the Einstein equation for a FLRW spacetime with a perfect fluid, we obtain the well known Friedmann equations

$$\begin{aligned} H^2 + \frac{K}{a^2} &= \frac{2}{(d-2)(d-1)}\left(8\pi\rho + \Lambda\right) \\ \frac{\ddot{a}}{a} &= -\frac{8\pi}{d-2}\left(\frac{d-3}{d-1}\rho + p\right) + \frac{2}{(d-2)(d-1)}\Lambda. \end{aligned} \tag{2.8}$$

To obtain the pure de Sitter form of these equations, we simply set $p = \rho = 0$, resulting in

$$\begin{aligned} H^2 + \frac{K}{a^2} &= \frac{2}{(d-2)(d-1)}\Lambda \\ \frac{\ddot{a}}{a} &= \frac{2}{(d-2)(d-1)}\Lambda, \end{aligned} \tag{2.9}$$

from which we see that the Hubble parameter has a constant value in the de Sitter case and that our universe expands (contracts) exponentially for $\Lambda > 0$ ($\Lambda < 0$). In the de Sitter case the different values of K describe the same physical spacetime in different coordinate systems [1]. We can therefore set $K = 0$. Then the Einstein equation gives the relation,

$$\Lambda = \frac{(d-2)(d-1)}{2} H^2. \tag{2.10}$$

For this it is apparent that $H = \dot{a}/a$ is an exact constant for de Sitter space and only depends on the energy content our our universe which is given by Λ.

[1] The notion of positive and negative curvature are conventional and depend on the choice of the metric signature and the definition of the Riemann tensor. We have taken the convention that positive curvature is equivalent to a negative Ricci scalar.

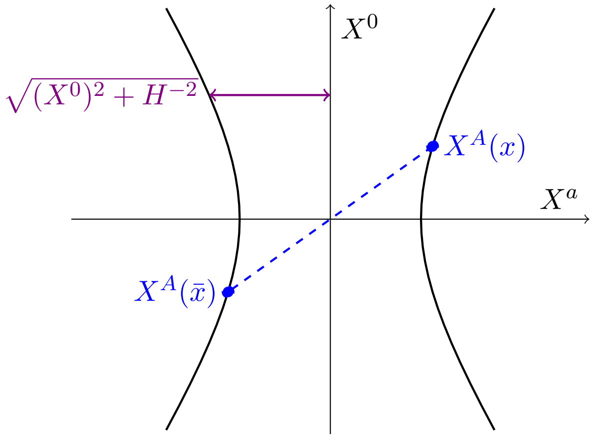


Figure 2.1: A simplified graphical representation of the embedded hyperboloid described by eq. 2.11. The spatial dimensions $\{X^a\}_{a=1,\dots,D-1}$ are represented together on the horizontal axis. Every point in time, X^0, represents a $(D-1)$-sphere of radius $\sqrt{(X^0)^2 + H^{-2}}$. Also visualised are a point $X^A(x)$ and its antipodal complement $X^A(\bar{x})$.

2.2 The geometric definition

De Sitter space in d dimensions, dS^d, can be embedded as a hyperboloid in $D = d+1$ dimensional Minkowski space $\mathcal{M}^D$ with coordinates $\{X^A\}_{A=0,\dots,D-1}$,

$$(X^0)^2 - (X^1)^2 - \dots - (X^D)^2 = \eta_{AB} X^A X^B = -H^{-2}, \quad A, B = 0, \dots, D-1, \tag{2.11}$$

where $\eta_{AB} = \mathrm{diag}(1, -1, \dots, -1)$ and H is just some constant which will turn out to equal the Hubble parameter. By analytic continuation $X^0 \to iX^0$, eq. 2.11 can be related to a D dimensional sphere. Furthermore, eq. 2.11 describes a $(D-1)$-sphere with radius $\sqrt{(X^0)^2 + H^{-2}}$, as shown in fig. 2.1.

Let us now investigate the further geometric properties resulting from this embedding condition. We start with computing an expression for the metric by taking the differential of $\eta_{\alpha\beta} X^\alpha X^\beta - (X^D)^2 = -H^{-2}$, to find

$$dX^D = \frac{\eta_{\alpha\beta} X^\alpha dX^\beta}{X^D} = \pm \frac{\eta_{\alpha\beta} X^\alpha dX^\beta}{\sqrt{\eta_{\mu\nu} X^\mu dX^\nu + H^{-2}}}, \tag{2.12}$$

where $\alpha, \beta = 0, \dots, d-1$. Moreover, we can define new coordinates $\{x^\alpha\}_{\alpha=0,\dots,d-1}$ on dS^d with a line element

$$ds^2 = \eta_{AB}\, dX^A\, dX^B = g_{\alpha\beta}\, dx^\alpha\, dx^\beta. \tag{2.13}$$

The metric $g_{\alpha\beta}$ is then given by

$$g_{\alpha\beta} = \eta_{\alpha\beta} - \frac{X_\alpha X_\beta}{\eta_{\mu\nu} X^\mu X^\nu + H^{-2}} \tag{2.14}$$

which can be easily inverted to

$$g^{\alpha\beta} = \eta^{\alpha\beta} + H^2 X^\alpha X^\beta. \tag{2.15}$$

Furthermore, for the Christoffel symbols one finds

$$\Gamma^{\alpha}_{\beta\gamma} = -H^2 \left(X^{\alpha}\eta_{\beta\gamma} - \frac{X^{\alpha}X_{\beta}X_{\gamma}}{\eta_{\mu\nu}X^{\mu}X^{\nu} + H^{-2}} \right). \tag{2.16}$$

Using these results, we can directly calculate the Ricci tensor as

$$R_{\alpha\beta} = -(d-1)H^2 g_{\alpha\beta} = -\frac{2}{d-2}\Lambda g_{\alpha\beta}, \tag{2.17}$$

in agreement with the previous result, eq. 2.6, if we interpret H as the Hubble parameter. By contracting with the above metric, we find the Ricci scalar

$$R = -d(d-1)H^2 = -\frac{2d}{d-2}\Lambda, \tag{2.18}$$

which also agrees with the previous result, eq. 2.5. Therefore we conclude, that eq. 2.11 is indeed a solution for eq. 2.10. Since dS^d is a maximally symmetric space, the Riemann tensor is given by [10, 11],

$$R_{\alpha\beta\gamma\delta} = \frac{R}{d(d-1)} \left(g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta\gamma} \right), \tag{2.19}$$

determined fully in terms of the Ricci scalar and the metric. This further leads to the identities

$$R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} = \frac{2R^2}{d(d-1)}, \quad \text{and} \quad R_{\alpha\beta}R^{\alpha\beta} = \frac{R^2}{d}. \tag{2.20}$$

2.3 Symmetries and the geodesic distance

As already noted, de Sitter spacetime is maximally symmetric, meaning that it possesses the same number of symmetries as Minkowski space. We observe that eq. 2.11 is invariant under the action of the full de Sitter group $O(1,d)$. $O(1,d)$ is also referred to as Poincaré or inhomogeneous Lorentz group in the literature. For a general element $\Omega \in O(1,d)$, there are four disconnected components characterised by $\det\Omega = \pm 1$ and ${\Omega^0}_0 \geq 1$ or ${\Omega^0}_0 \leq -1$. Let $SO(1,d)$ be the component containing the identity element,

$$SO(1,d) \coloneqq \{\Omega \in O(1,d) \,|\, \det\Omega = +1,\, {\Omega^0}_0 \geq 1\}. \tag{2.21}$$

In this subgroup the direction of time and parity are conserved. Time reversal and spatial reflection can be characterised by

$$\begin{aligned} T &= \mathrm{diag}(-1,1,\dots,1) \quad &\in O(1,d), \\ P &= \mathrm{diag}(1,-1,1,\dots,1) \quad &\in O(1,d) \end{aligned} \tag{2.22}$$

respectively [4]. Using these elements, we can easily define the other three components of $O(1,d)$, which contain elements connected to

- time reversal,

$$O_T(1,d) \coloneqq \{\Omega \cdot T \,|\, \Omega \in SO(1,d)\} = \{\Omega \in O(1,d) \,|\, \det\Omega = -1,\, {\Omega^0}_0 \leq -1\}, \tag{2.23}$$

- spatial reflection,

$$O_P(1,d) \coloneqq \{\Omega \cdot P \,|\, \Omega \in SO(1,d)\} = \{\Omega \in O(1,d) \,|\, \det\Omega = -1,\, {\Omega^0}_0 \geq 1\}, \tag{2.24}$$

- and spacetime reflection,

$$O_{TP}(1,d) \coloneqq \{\Omega \cdot T \cdot P \,|\, \Omega \in SO(1,d)\} = \{\Omega \in O(1,d) \,|\, \det\Omega = 1,\, {\Omega^0}_0 \leq -1\}. \tag{2.25}$$

For every point $x \in dS^d$ or $X^A(x) \in \mathcal{M}^{d+1}$ there exists an antipodal point $\bar{x} \in dS^d$ or $X^A(\bar{x}) \in \mathcal{M}^{d+1}$. These related by the transformation

$$A = \mathrm{diag}(-1,-1,\ldots,-1) \quad \in O_T(1,d), \tag{2.26}$$

which is an element of $O_T(1,d)$ for even d. Therefore for antipodal points $X^A(x) = -X^A(\bar{x})$ [4]. These are visualised in fig. 2.1.
The geodesic distance between two spacetime points x and y connected by a geodesic with affine parameter λ is defined as [4]

$$d(x,y) = \int_x^y \sqrt{\eta_{\mu\nu}\dot{X}^\mu \dot{X}^\nu} d\lambda. \tag{2.27}$$

By complete analogy with the sphere, the geodesic distance for de Sitter geometry can be written as [8] [2]

$$d(x,y) = \frac{1}{H} \arccos Z(x,y). \tag{2.28}$$

where we have defined a function of spacetime points x and y,

$$Z(x,y) \coloneqq -H^2 \eta_{AB} X^A(x) X^B(y), \tag{2.29}$$

which is symmetric in x and y and under antipodal transformations $Z(x,y) = -Z(\bar{x},y)$. Hence Z alone does not distinguish between past and future light cones [18]. The function $Z(x,y)$ can also be written in terms of the squared distance between two point in the embedding space, $X^\mu(x)$ and $X^\mu(y)$, as

$$\begin{aligned} Z(x,y) - 1 &= \frac{H^2}{2}\left(X^A(x) - X^A(y)\right)^2, \\ Z(x,y) + 1 &= -\frac{H^2}{2}\left(X^A(x) - X^A(\bar{y})\right)^2. \end{aligned} \tag{2.30}$$

From these results we can determine bounds on $Z(x,y)$, depending on the separation of x and y. The results are summarised in fig. 2.2. We see that if x is within the light cone of y, $Z > 1$ and $Z \to 1$ as x approaches the boundaries of the light cone. For $Z < -1$ no geodesic exists that joins $X^A(x)$ and $X^A(y)$. For antipodal points $Z = H^2 \eta_{AB} X^A(x) X^B(x) = -1$, which therefore just remain in causal contact with each other. There are two geodesics connecting a point with its antipodal point. This can be seen from fig. 2.1, imagining that we can move in either direction around the hyperboloid.

[2] The geodesic distance d on a sphere of radius r is $d = r\theta$, where θ is the angle separating two points $\boldsymbol{x}$ and $\boldsymbol{y}$ on the surface of the sphere. Hence, in Euclidean space we have $\boldsymbol{x} \cdot \boldsymbol{y} = r^2 \cos\theta = r^2 \cos(d/R)$.

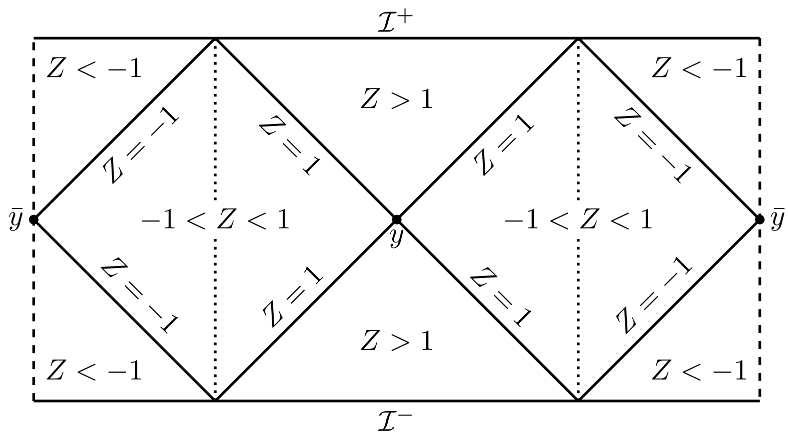


Figure 2.2: The values of $Z(x, y)$ in certain regions of space, for fixed y. The dashed vertical lines on the far left and right are to be identified with each other. Halfway between y and $\bar{y}$, on the vertical dotted lines, $Z = 0$ [4].

2.4 Coordinate patches on the de Sitter spacetime

To do actual physically relevant calculations, it is very useful to specify different coordinate systems covering our geometry. Below we will quickly review the most relevant de Sitter coordinate patches. A more extensive discussion can be found e.g. in [1].

2.4.1 The global patch

Global coordinates are the solution of eq. 2.11 given by [8, 11]

$$\begin{aligned} X^0 =& \frac{1}{H}\sinh(Ht), \\ X^i =& \frac{1}{H}\cosh(Ht)\sin\theta_1\ldots\sin\theta_{i-1}\cos\theta_i, \\ X^D =& \frac{1}{H}\cosh(Ht)\sin\theta_1\ldots\sin\theta_i, \end{aligned} \tag{2.31}$$

where $i = 1, \ldots, D-1$. In these coordinates the induced metric becomes

$$ds^2 = dt^2 - \frac{\cosh^2(Ht)}{H^2}d\Omega^2_{D-2}. \tag{2.32}$$

These coordinates cover the complete manifold and hence the topology of de Sitter space is $\mathbb{R}\times S^{D-2}$.
Comparing with the FLRW case, as defined in eq. 2.1, here

$$a(t) = \frac{\cosh(Ht)}{H}, \tag{2.33}$$

which corresponds to the $K = 1$ FLRW solution. Hence de Sitter space describes a spatial sphere with radius $a(t)$. It contracts from $a(-\infty) = \infty$ to $a(\infty) = \infty$, reaching its minimum at $a(0) = \frac{1}{H}$. In the inflationary paradigm these coordinates are less useful, as the contracting phase has no place. In that case one generally turns to Poincaré coordinates, discussed in the

next section.
The conformal time, defined dy $d\eta = dt/a(t)$ is given by

$$\eta = 2\arctan\left(\tanh\left(\frac{Ht}{2}\right)\right), \tag{2.34}$$

and covers the range $-\frac{\pi}{2} < \eta < \frac{\pi}{2}$.
Lastly, we want to calculate $Z(x, y)$ in these coordinates via eq. 2.29, resulting in

$$\begin{aligned} Z = \cosh(Ht_x)\cosh(Ht_y)\big(&\cos\theta_1\cos\phi_1 \\ &+ \sin\theta_1\sin\phi_1\big(\cos\theta_2\cos\phi_2 \\ &+ \sin\theta_2\sin\phi_2\big(\cos\theta_3\cos\phi_3 \\ &+ \dots \\ &+ \sin\theta_{D-4}\sin\phi_{D-4}\big(\cos\theta_{D-3}\cos\phi_{D-3} \\ &+ \sin\theta_{D-3}\sin\phi_{D-3}\cos(\theta_{D-2} - \phi_{D-2})\big)\dots\big) \\ &- \sinh(Ht_x)\sinh(Ht_y), \end{aligned} \tag{2.35}$$

where $x = (t_x, \theta_1, \dots)$ and $x = (t_x, \phi_1, \dots)$. In global coordinates Z has a quite complicated form, but still obeys the bounds in 2.2. We will see that in Poincaré coordinates we will have a much simpler expression for Z.

2.4.2 The expanding and contracting Poincaré patches

Poincaré coordinates are the solution of eq. 2.11 given by [8]

$$\begin{aligned} X^0 =& \frac{1}{H}\sinh Ht + \frac{H}{2}e^{Ht}(x^i)^2, \\ X^i =& x^i e^{Ht}, \\ X^D =& -\frac{1}{H}\cosh Ht + \frac{H}{2}e^{Ht}(x^i)^2, \end{aligned} \tag{2.36}$$

where $i = 1, \dots, D-1$. In these coordinates the induced metric becomes

$$ds^2 = dt^2 - e^{2Ht}\delta_{ij}dx^i dx^j, \tag{2.37}$$

The Poincaré patch correspond to a flat ($K = 0$) FLRW universe with $a(t) = e^{Ht}$, describing exponentially expanding spatial slices, or in other words, an exponentially expanding universe. In this case it is especially useful to introduce a conformal time,

$$\eta = \int dt\, e^{-Ht} = -\frac{1}{H}e^{-Ht}, \tag{2.38}$$

so that the metric becomes

$$ds^2 = \frac{1}{H^2\eta^2}\left(d\eta^2 - \delta_{ij}dx^i dx^j\right), \tag{2.39}$$

with $a(\eta) = \frac{1}{H^2\eta^2}$. We see that the range $-\infty < t < \infty$ corresponds to $-\infty < \eta < 0$. These coordinates cover only half of the complete manifold, as can be seen from $X^0 - X^D = \frac{1}{H}e^{Ht}$ which implies $X^0 \geq X^D$. Consequently this patch is usually referred to as the expanding

Poincaré patch. These coordinates are most often used in inflationary cosmology as they only describe the expanding part of the spacetime, which is a good approximation for the inflationary stage. The other half of our manifold is reached by letting $t \to -t$. Here $a(t) = e^{-Ht}$ and therefore this is usually referred to as contracting Poincaré patch.
Using eqs. 2.36 we can easily calculate $Z(x, y)$ via eq. 2.29,

$$Z(x, y) = 1 + \frac{(\eta_x - \eta_y)^2 - (\boldsymbol{x} - \boldsymbol{y})^2}{2\eta_x\eta_y}. \tag{2.40}$$

This has a much nicer form than above and we will encounter it again and again in the forthcoming sections.

2.4.3 The static patch

There is another set of coordinates, which are referred to as static coordinates. They are given by the set [12]

$$\begin{aligned} X^0 =& \frac{1}{H}\sqrt{1 - (rH)^2}\sinh(Ht), \\ X^i =& r\sin\theta_1 \ldots \sin\theta_{i-1}\cos\theta_i, \\ X^D =& \frac{1}{H}\sqrt{1 - (rH)^2}\cosh(Ht), \end{aligned} \tag{2.41}$$

where $i = 1, \ldots, D - 1$. The metric takes the form

$$ds^2 = (1 - (Hr)^2)dt^2 - \frac{dr^2}{1 - (Hr)^2} - r^2 d\Omega_{D-3}. \tag{2.42}$$

Horizons in these coordinates correspond to $r = 1/H$ and $t = \pm\infty$, where the coordinates become singular. We can extend them to $r > H$, but then t and r flip their respective roles – t becomes spacelike and r becomes timelike.
These coordinates are interesting, because they are the only set of coordinates, in which the metric takes a time-independent form. The implications of this will be discussed in the next section.

2.5 Timelike Killing vector fields

The notion of a timelike Killing vector is essential for energy conservation, choosing a vacuum state and therefore having a clear notion of particles in our spacetime. Here we want to investigate timelike Killing vectors in de Sitter space.
By looking at the metrics eq. 2.32 and eq. 2.39, we see that both of them carry explicit time dependence and therefore neither ∂_t nor ∂_η are Killing vectors in these coordinates.
But now consider the metric of the static patch, eq. 2.42. Here we have no time dependence at all and therefore ∂_t represents a Killing vector. The problem here is that at the horizon $r = 1/H$ the coordinates become singular and the norm of ∂_t vanishes. Hence the spacetime is essentially cut into four pieces, which can be seen from the conformal diagram in fig. 2.3. Also indicated on the diagram is the flow of the Killing vector field ∂_t. Hence there is no globally timelike Killing vector in de Sitter space [12]. Consequently, there is no notion of energy conservation, no global vacuum state and therefore no global notion of particles. We

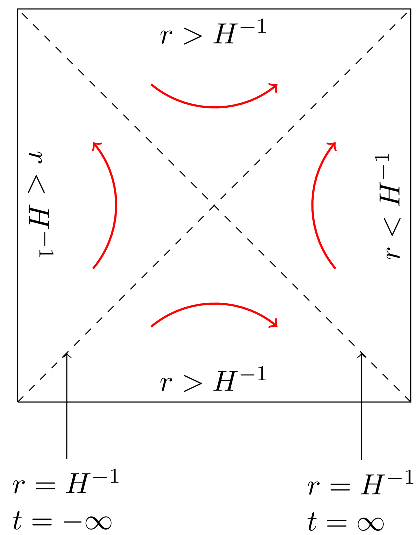


Figure 2.3: The conformal diagram of the static de Sitter patch. The space is split into four wedges by the horizons. The arrows indicate the flow of the Killing vector field ∂_t.

base most physical interpretations in quantum field theory nowadays on the concept of energy and a unique Fock vacuum state. Now we have to pay a high price for the lack of energy conservation and hence the missing notion of a vacuum state.

An interesting approach would be to try and build a theory around the existing symmetries of a given spacetime. The usual steps to follow would be to construct the invariant Casimir operators out of the symmetry group generators. In a flat spacetime we generally have Poincaré invariance – in particular time translational symmetry, which is responsible for global energy conservation. In de Sitter spacetime we are missing exactly this time translational symmetry and hence violate energy conservation. Taking this path, one finds that the group theoretic approach leads to the same scalar field equation of motion as if we would construct an invariant scalar action and take its variation. The latter is the approach we will take in sec. 3. A nice and short discussion of the other path is given for example in [13]. The main point is that the quadratic invariant Casimir operator of $SO(1, 4)$ in the Poincaré patch takes the exact same form as the differential operator derived from the variation of the action.

Chapter 3

Scalar fields in a de Sitter background

In this section we want to, first of all, point out the steps to generalize the action of any field in Minkowski space to any other spacetime geometry in sec. 3.1 [14]. Then we proceed to analyse the action we constructed for a scalar field first in a general FLRW spacetime (sec. 3.2 [14, 15]) and discuss specific coupling terms in sec. 3.3 [1, 14]. Then we specify to de Sitter spacetime. We discuss how to quantise our scalar field in the expanding Poincaré patch, solve for the mode functions (sec. 3.4 [1, 4, 8, 16–21]) and discuss the vacuum ambiguity by considering the diagonalisation of the Hamiltonian. Furthermore we construct the two point function for the euclidean vacuum, which corresponds to one particular vacuum choice (sec. 3.5 [4, 8, 18]) explicitly. Lastly, we generalise the two point function for any physically sensible alternative choice of vacuum in sec. 3.6 [7] and then construct other Green functions from the two point function, discuss de Sitter invariance and what this means for different vacuum choices 3.7 [4, 18].

3.1 Generalizing actions to curved spacetime

In flat Minkowski spacetime the action of a free scalar field is given by

$$S[\phi] = \frac{1}{2}\int d^dx\left(\eta^{\mu\nu}\partial_\mu\phi(x)\partial_\nu\phi(x) - m^2\phi(x)^2\right). \tag{3.1}$$

If our goal is to generalise this to curved spacetime we require general covariance. To achieve this we must preform the following changes [14],

- Replace the Minkowski metric by our general spacetime metric, $\eta_{\mu\nu} \to g_{\mu\nu}$.
- Replace standard derivatives by covariant derivatives, $\partial_\mu \to \nabla_\mu$.
- Make the integral measure in the action covariant by replacing $d^dx \to d^dx\sqrt{|g|}$, where $g = \det g_{\mu\nu}$.

Following these steps, we can write down the action of a massive scalar field with a general curved background

$$S[\phi, g_{\mu\nu}] = \frac{1}{2}\int d^dx\sqrt{|g(x)|}\left[g^{\mu\nu}\nabla_\mu\phi(x)\nabla_\nu\phi(x) - m^2\phi^2(x)\right], \tag{3.2}$$

where the action is now generally covariant and of course now has functional dependence on $g_{\mu\nu}$.

3.2 Free scalar fields in a FLRW universe

Having constructed the scalar field action for a general spacetime, let us keep things general as long as possible and start by analysing a free scalar field in a d-dimensional FLRW spacetime, with metric

$$ds^2 = a^2(\eta)\left(d\eta^2 - \delta_{ij}dx^i dx^j\right), \quad i,j = 1,\ldots,d-1. \tag{3.3}$$

Later we can specialize to the de Sitter case, by specifying the form of the scale factor $a(\eta)$. We start from the action of a scalar field, eq. 3.2 which we constructed above,

$$S[\phi, g_{\mu\nu}] = \frac{1}{2}\int d^dx\,\sqrt{|g(x)|}\left[g^{\mu\nu}\nabla_\mu\phi(x)\nabla_\nu\phi(x) - m^2\phi^2(x)\right]. \tag{3.4}$$

Substituting the above metric, integrating by parts and using $\sqrt{|g|} = a^d$,

$$S[\phi, a] = -\frac{1}{2}\int d^dx\,a^d\phi(x)\left[\Box + m^2\right]\phi(x). \tag{3.5}$$

The standard approach in flat space quantum field theory is to canonically normalize fields to obtain a kinetic term of the form

$$\frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi \quad \text{or equivalently} \quad -\frac{1}{2}\phi\Box_M\phi, \tag{3.6}$$

where $\Box_M \coloneqq \eta^{\mu\nu}\partial_\mu\partial_\nu$ is the Minkowski d'Alembertian operator. In the above action,

$$-\frac{1}{2}a^d\phi\Box\phi = -\frac{a^{d-2}}{2}\phi\Box_M\phi - \frac{(d-2)}{2}\frac{a'a^d}{a^3}\phi\phi', \tag{3.7}$$

which does not resemble a canonically normalised kinetic term in the action. But if we define a new auxiliary field $\chi(x) = a^{\frac{d-2}{2}}\phi(x)$ [14, 15], we find that

$$-\frac{1}{2}a^d\phi\Box\phi = -\frac{1}{2}\chi\Box_M\chi + \frac{d-2}{8}\left((d-4)\left(\frac{a'}{a}\right)^2 + 2\frac{a''}{a}\right)\chi^2. \tag{3.8}$$

This gives the correct canonically normalised term, but with the price of introducing a time dependent mass-like term. We can now transform to partial Fourier space,

$$\chi(x) = \int đ^{d-1}k\,\chi_{\boldsymbol{k}}(\eta)\,e^{i\boldsymbol{k}.\boldsymbol{x}}, \tag{3.9}$$

where we only transform the spatial coordinates, due to the η dependence of the scale factor in the metric. The definition of the integral measure $đ^{d-1}k$ is given in app. A. Then the full action in partial Fourier space, in terms of mode functions, becomes

$$S_{\boldsymbol{k}',\boldsymbol{k}}[\chi, a] = -\frac{1}{2}\int d^dx\,\chi_{\boldsymbol{k}'}\left[\partial_\eta^2 + k^2 + m^2a^2 - \frac{d-2}{4}\left((d-4)\left(\frac{a'}{a}\right)^2 + 2\frac{a''}{a}\right)\right]\chi_{\boldsymbol{k}}. \tag{3.10}$$

This resembles a harmonic oscillator with time dependent frequency, which is essentially a very general feature of quantum field theory in curved spacetime.

We must at this point stress that taking the limits of low or high momenta is not as simple as in flat space. In flat space we usually take the scalar field mass as the given energy scale and

then compare k^2/m^2 to define what we mean by ultraviolet (UV: $k/m \to \infty$) and infrared (IR: $k/m \to 0$). As soon as we move within a dynamic background we have the complication that the quantities we should compare are $k^2/(ma)^2$ and as a changes, our definitions of UV and IR change simultaneously. To this end we can define the *comoving momentum* k/a, which is a scale that does not change with the expansion of the universe.
For example in the expanding Poincaré patch of de Sitter we have $a^2 = (H\eta)^{-2}$ and hence

$$\frac{k^2}{a^2m^2} = (k\eta)^2 \frac{H^2}{m^2}, \tag{3.11}$$

which sets the energy scale. So when we talk about the IR or UV limit in this patch of de Sitter space, what we mean is that $(k\eta)^2 \ll m^2/H^2$ or $(k\eta)^2 \gg m^2/H^2$ respectively. This then allows us to drop either the second or third them in eq. 3.10. Therefore, we have seen that defining UV and IR regimes in curved, dynamic spacetimes is not quite as simple as for the static cases.

3.3 The addition of interaction terms

So far the effect of a curved background was only due to the generalisation of the integral measure and the introduction of covariant derivatives. But we can also explicitly couple our field to the background, by introducing an interaction term,

$$S[\phi, g_{\mu\nu}] = \frac{1}{2}\int d^dx \sqrt{|g(x)|}\left[g^{\mu\nu}\nabla_\mu\phi(x)\nabla_\nu\phi(x) - m^2\phi^2(x) - \xi R\phi^2(x)\right], \tag{3.12}$$

which allows for a dimensionless coupling parameter ξ. In a general dimensional FLRW spacetime the Ricci scalar is given by

$$R = -(d-1)\frac{(d-4)a'^2 + 2aa''}{a^4}. \tag{3.13}$$

Adding the $\xi R\phi^2$ term to eq. 3.10 and substituting the above result, we find

$$\begin{aligned} &S_{\boldsymbol{k}',\boldsymbol{k}}[\chi, a] = \\ &= -\frac{1}{2}\int d^dx\, \chi_{\boldsymbol{k}'}\left[\partial_\eta^2 + k^2 + m^2a^2 - (d-1)\left(\frac{d-2}{4(d-1)} - \xi\right)\left((d-4)\left(\frac{a'}{a}\right)^2 + 2\frac{a''}{a}\right)\right]\chi_{\boldsymbol{k}}. \end{aligned} \tag{3.14}$$

So we see that if our theory is *conformally coupled*, $\xi = \frac{d-2}{4(d-1)}$, the only term where spacetime curvature changes the form of the action is in the mass term. This is expected since the mass fixes an energy scale and is hence not conformally (scale) invariant. Whereas spacetimes such as FLRW are conformally flat, $g_{\mu\nu} = a^2(\eta)\eta_{\mu\nu}$, and hence a conformally invariant theory with conformal coupling leaves the action completely invariant.
On the other hand, setting $\xi = 0$ is known as *minimal coupling*. Minimal, as the curvature effects are still manifest in the action due to the determinant of the metric.
For $d = 4$ the term with first order derivative of the scale factor disappears and the action simplifies to

$$S_{\boldsymbol{k}',\boldsymbol{k}}[\chi, a] = -\frac{1}{2}\int d^dx\, \chi_{\boldsymbol{k}'}\left[\partial_\eta^2 + \left[k^2 + m^2a^2 - (1-6\xi)\frac{a''}{a}\right]\right]\chi_{\boldsymbol{k}}. \tag{3.15}$$

The equation of motion resulting from this action for a minimally coupled field,

$$\left[\partial_\eta^2 + k^2 + m^2 a^2 - \frac{a''}{a}\right]\chi_{\boldsymbol{k}} = 0 \tag{3.16}$$

is known as the *Mukhanov-Sasaki* equation [1, 14].
Summarizing, we have established that explicit mass terms break conformal invariance as they fix a certain energy scale, which in return is not invariant under conformal transformations. Additionally, we have found that conformal coupling of a free, massless scalar theory decouples from gravity completely.
If we add interaction terms, which contain higher powers ($n > 2$) of fields we have terms of the form

$$S[\chi, a] = \int d^d x \left(\ldots + a^d \frac{\lambda}{n}\phi^n(x)\right) = \int d^d x \left(\ldots + a^{d-\frac{n}{2}(d-2)}\frac{\lambda}{4}\chi^n(x)\right) \tag{3.17}$$

in the action, which again couples to gravity. But also here, for the special case when $n = \frac{2d}{d-2}$, the interaction terms decouple from gravity. This is the case for example for $\lambda\phi^4$ theory in $d = 4$ dimensions.

3.4 Quantising scalar fields in the expanding Poincaré patch

In this section we want to compute the mode functions of a scalar field in the Poincaré patch of dS^d. For the expanding Poincaré patch we have $a^2 = 1/(H\eta)^2$, where η has the range $-\infty < \eta < 0$ as $-\infty < t < \infty$. But since $a > 0$ we will, for simplicity, always consider the magnitude of η here and redefine $\eta : \infty \to 0$ as $t : -\infty \to \infty$. We start from the action of a scalar field in a FLRW universe, given by eq. 3.10, which after substituting for $a(\eta)$ becomes

$$S_{\boldsymbol{k}',\boldsymbol{k}}[\chi, a] = -\frac{1}{2}\int d^d x\, \chi_{\boldsymbol{k}'}\left[\partial_\eta^2 + k^2 + \frac{m^2}{(H\eta)^2} - \frac{d(d-2)}{4\eta^2}\right]\chi_{\boldsymbol{k}}. \tag{3.18}$$

We have not included any coupling terms such as $\xi R\chi^2$, but one could easily generalize this to the coupled case by generalising m^2 to the effective mass $M^2 = m^2 - \zeta R + \ldots$, i.e. the mass term of our scalar field plus additional (constant) terms describing coupling to curvature. This can be done so easily because for de Sitter $R = const.$, from eq. 2.5.
We have already expanded the field χ in partial Fourier modes,

$$\chi(x) = \int đ^{d-1}k \chi_{\boldsymbol{k}}(\eta) e^{i\boldsymbol{k}.\boldsymbol{x}}. \tag{3.19}$$

Since the field ϕ is real, so is the auxiliary field χ, the mode functions must obey $\chi^*_{\boldsymbol{k}} = \chi_{-\boldsymbol{k}}$. We see from eq. 3.10, that the differential operator only depends on $k = |\boldsymbol{k}|$ and so the general expansion of the mode functions is [1]

$$\chi_{\boldsymbol{k}}(\eta) = \frac{1}{\sqrt{2}}\left(a_{\boldsymbol{k}} q^*_k(\eta) + a^\dagger_{-\boldsymbol{k}} q_k(\eta)\right), \tag{3.20}$$

where q_k and q^*_k are linearly independent solutions to the equation on motion and as $\chi_{\boldsymbol{k}}$ is real, $a^\dagger_{\boldsymbol{k}}$ and $a_{\boldsymbol{k}}$ are conjugate operators. Let us consider these creation and annihilation operators to define some arbitrary Fock space for now. Upon substituting this mode expansion into eq. 3.18, we see that in the Poincaré patch, the equation of motion for the mode functions becomes

$$\left[\eta^2\partial_\eta^2 + (k\eta)^2 + \frac{m^2}{H^2} - \frac{d(d-2)}{4}\right] q_k = 0. \tag{3.21}$$

This equation is the Bessel function in disguise, with the solutions

$$q_k(\eta) = \sqrt{\frac{\pi\eta}{2}}\left(A_1 H_n^{(1)}(k\eta) + A_2 H_n^{(2)}(k\eta)\right), \tag{3.22}$$

where $n := \sqrt{\frac{(d-1)^2}{4} - \frac{m^2}{H^2}}$, $H_n^{(1,2)}(k\eta)$ are the standard Hankel functions and A_1, A_2 are complex coefficients. By imposing the (equal time) commutation relations of the field χ with the canonical momentum $\pi = \frac{\delta S}{\delta \chi'} = \chi'$, where $(\,.\,)' := \frac{\partial\,.}{\partial\eta}$,

$$\begin{aligned} [\chi(x), \chi(y)]_{x^0=y^0} &= [\pi(x), \pi(y)]_{x^0=y^0} = 0, \\ [\chi(x), \pi(y)]_{x^0=y^0} &= i\delta^{(d-1)}(\boldsymbol{x}-\boldsymbol{y}), \end{aligned} \tag{3.23}$$

or equivalently for the operators $a_{\boldsymbol{k}}$ and $a_{\boldsymbol{k}}^\dagger$

$$\begin{aligned} \left[a_{\boldsymbol{k}}, a_{\boldsymbol{k'}}\right] &= \left[a_{\boldsymbol{k}}^\dagger, a_{\boldsymbol{k'}}^\dagger\right] = 0, \\ \left[a_{\boldsymbol{k}}, a_{\boldsymbol{k'}}^\dagger\right] &= \delta^{(d-1)}(\boldsymbol{k}-\boldsymbol{k'}), \end{aligned} \tag{3.24}$$

we can also fix the normalisation of our mode functions. In order to obey the above commutation relations, we find that the mode functions must satisfy

$$q_k(q_k^*)' - (q_k)'q_k^* = W[q_k, q_k^*] = -2i, \tag{3.25}$$

where $W[.,.]$ is the Wronskian. Note that for $W[q_k, q_k^*] \neq 0$ the solutions q_k and q_k^* are linearly independent. For the solution eq. 3.22 we find the normalisation condition

$$|A_1|^2 - |A_2|^2 = 1. \tag{3.26}$$

Therefore we can parametrise these coefficients as

$$A_1 = \cosh\alpha \quad \text{and} \quad A_2 = \sinh\alpha\; e^{i\beta}, \tag{3.27}$$

up to an overall phase and the general mode function becomes

$$q_k^{(\alpha,\beta)}(\eta) = \sqrt{\frac{\pi\eta}{2}}\left(\cosh\alpha\; H_n^{(1)}(k\eta) + \sinh\alpha\; e^{i\beta}\; H_n^{(2)}(k\eta)\right). \tag{3.28}$$

We see that we have a two parameter freedom for the choice of our mode function. In principle we should now check if diagonalisation of the Hamiltonian fixes any of these parameters further. To this end, let us consider the Hamiltonian of the free theory for our auxiliary field,

$$\int d\eta\, H(\eta) = \int d^dx\, \pi\partial_\eta\chi - S[\chi, g_{\mu\nu}], \tag{3.29}$$

where $S[\chi, g_{\mu\nu}]$ is the action of our auxiliary field from eq. 3.10. Upon substituting the mode expansion from eq. 3.20, the Hamiltonian becomes

$$\begin{aligned} H(\eta) = \int đ^{d-1}k\, \frac{1}{2}\Bigg[& \left(\left|q_k^{(\alpha,\beta)\prime}\right|^2 + \omega(\eta)^2 \left|q_k^{(\alpha,\beta)}\right|^2\right) a_{\boldsymbol{k}}^\dagger a_{\boldsymbol{k}} \\ & + \left(q_k^{(\alpha,\beta)\prime 2} + \omega(\eta)^2 q_k^{(\alpha,\beta)2}\right) a_{\boldsymbol{k}}^\dagger a_{-\boldsymbol{k}}^\dagger + h.c.\Bigg], \end{aligned} \tag{3.30}$$

$$\text{where} \quad \omega(\eta)^2 = k^2 + m^2 a^2 - (d-1)\left(\frac{d-2}{4(d-1)} - \xi\right)\left((d-4)\left(\frac{a'}{a}\right)^2 + 2\frac{a''}{a}\right).$$

Essentially all complications arise because $\omega(\eta)^2$ has explicit time dependence through the scale factor. The second term is the problematic one as they mix Fock states of different occupation number. The Hamiltonian would be diagonalised at all times, if

$$q_k^{(\alpha,\beta)\prime 2} + \omega(\eta)^2 q_k^{(\alpha,\beta)2} = 0 \quad \forall\, \eta. \tag{3.31}$$

The formal solution to this equation is proportional to

$$q_k^{(\alpha,\beta)} \propto e^{\pm i \int^\eta d\tau\, \omega(\tau)}, \tag{3.32}$$

but this is not equal to the solution we found in eq. 3.28. Therefore, one would have to newly diagonalize the Hamiltonian at each value of η [8]. This is the point where we have to give up the idea of a global Fock space. In flat space, the plane wave solution solves the equation of motion and to the above equation simultaneously and hence the Hamiltonian can be diagonalised for all times.

The best we can do is investigate if we can diagonalise the Hamiltonian in the asymptotic past or future. Since our mode functions depend on the product $k\eta$ only, up to a multiplicative factor, this is equivalent of investigating the regimes of large and small comoving momenta. With this in mind let us look at the limiting cases of our solution in eq. 3.28. For $k\eta \to \infty$, corresponding to our UV regime with large comoving momentum, the mode functions become to leading order

$$q_k^{(\alpha,\beta)}(\eta) = \frac{1}{\sqrt{k}}\left(\cosh\alpha\ e^{ik\eta} + i\sinh\alpha e^{i\beta} e^{-ik\eta} + \mathcal{O}\left((k\eta)^{-1}\right)\right), \quad k\eta \to \infty, \tag{3.33}$$

up to an overall phase. In this limit the mode functions behave as plane waves. The coefficients do not mix and we get separate behaviour for the two different solutions $H_n^{(1,2)}(k\eta)$, respectively.

For $k\eta \to 0$, corresponding to our IR limit with small comoving momenta the mode functions become

$$q_k^{(\alpha,\beta)}(\eta) = \sqrt{\eta}\left(B_1 (k\eta)^{-n} + B_2 (k\eta)^n + \mathcal{O}\left((k\eta)^{2-n}\right)\right), \quad k\eta \to 0, \tag{3.34}$$

where the coefficients are

$$\begin{aligned} B_1 &= -i\frac{2^n(\cosh\alpha - \sinh\alpha\ e^{i\beta})\Gamma(n)}{\sqrt{2\pi}} \quad \text{and} \\ B_2 &= \sqrt{\frac{\pi}{2}}\frac{(1+i\cot(\pi n))\cosh\alpha + (1-i\cot(\pi n))\sinh\alpha e^{i\beta}}{2^n\Gamma(n+1)} \end{aligned} \tag{3.35}$$

Here we see that the coefficients mix and to impose a specific behaviour in the IR limit, we must take a different linear combination of Hankel functions to kill either of B_1 or B_2. To get $B_1 = 0$ we simply set

$$\cosh\alpha = \sinh\alpha \; e^{i\beta} \quad \text{giving} \quad B_2 = \sqrt{\frac{\pi}{2}} \frac{2\sinh\alpha e^{i\beta}}{2^n\Gamma(n+1)}. \tag{3.36}$$

Alternatively, to set $B_2 = 0$ we must let

$$\cosh\alpha = \sinh\alpha \; e^{i\beta+2i\pi n} \quad \text{giving} \quad B_1 = i\frac{2^n\Gamma(n)(1-e^{2i\pi n})\sinh\alpha \; e^{i\beta}}{\sqrt{2\pi}}. \tag{3.37}$$

Recall that the physical field is given by $\phi = (H\eta)^{\frac{d-2}{2}}\chi$. Hence, for $m \neq 0$ the mode functions in the IR decay to zero as $\eta \to 0$, independently of n being real or imaginary. For real n they homogeneously decay to zero and for imaginary n they oscillate while decaying.
Now let us come back to the question of appropriate boundary conditions and Hamiltonian diagonalisation. For short distance scales in the UV regime, $\omega^2 \approx k$ in eq. 3.30 and therefore the Hamiltonian reduces to flat Minkowski case. This is expected, as the large scale curvature of the spacetime will become completely negligible in this limit. Therefore we expect the mode functions to agree with the Minkowski solutions in the UV, and we can impose the condition

$$q_k(\eta) = \frac{1}{\sqrt{k}} \; e^{ik\eta}, \quad k\eta \to \infty. \tag{3.38}$$

This condition is achieved by setting $\alpha = 0$ and defines the so called *Bunch-Davies (BD) modes* [8]. Imposing this boundary condition, the mode functions diagonalise the Hamiltonian in the UV. Therefore we get the correct UV behaviour and with it a notion of particles and the flat space QFT we are used to [8].
Since the UV limit corresponds to $k\eta \to \infty$ we can interpret is as "early time" limit and therefore the BD-vacuum is also referred to as the *in*-vacuum, diagonalising the Hamiltonian at "early time". Similarly we can define the *out*-vacuum, corresponding to the mode functions which diagonalise the Hamiltonian at "late times", corresponding to the IR limit $k\eta \to 0$. From eq. 3.34 and the discussion below we know that these mode functions always decay in the IR limit and hence we are not forced to set either B_1 or B_2 to zero to achieve diagonalisation of the Hamiltonian. Although the leading order contribution will come from the first term in eq. 3.34, and we can set B_2 to zero.
Therefore we find the full expression for the BD mode expansion of our field,

$$\begin{aligned} \chi(x) &= \sqrt{\frac{\pi\eta}{4}} \int đ^{d-1}k \left(a_{\boldsymbol{k}} H_n^{(1)*}(k\eta) e^{i\boldsymbol{k}.\boldsymbol{x}} + a_{\boldsymbol{k}}^\dagger H_n^{(1)}(k\eta) e^{-i\boldsymbol{k}.\boldsymbol{x}} \right), \quad \text{or} \\ \phi(x) &= (H\eta)^{\frac{d-1}{2}} \sqrt{\frac{\pi}{4H}} \int đ^{d-1}k \left(a_{\boldsymbol{k}} H_n^{(1)*}(k\eta) e^{i\boldsymbol{k}.\boldsymbol{x}} + a_{\boldsymbol{k}}^\dagger H_n^{(1)}(k\eta) e^{-i\boldsymbol{k}.\boldsymbol{x}} \right). \end{aligned} \tag{3.39}$$

This mode expansion now gives the correct UV behaviour, matching onto scalar quantum field theory in flat space in the limit of large momenta and/or negligible curvature. This consequently defines a specific vacuum $a_{\boldsymbol{k}}\big|0\big\rangle = 0$, which will be referred to as the Bunch-Davies vacuum.
But what about the other choices of (α, β)? For convenience, let us define

$$f_{\boldsymbol{k}}^{(\alpha,\beta)}(x) := q_{\boldsymbol{k}}^{(\alpha,\beta)}(\eta) e^{-i\boldsymbol{k}.\boldsymbol{x}}. \tag{3.40}$$

Having fixed one particular mode solution, we can express the general (α,β)-mode function as

$$f_{\boldsymbol{k}}^{(\alpha,\beta)}(x) = \cosh\alpha\, f_{\boldsymbol{k}}(x) + \sinh\alpha\; e^{i\beta} f_{\boldsymbol{k}}^{*}(x), \tag{3.41}$$

which defines a mode-independent *Bogoliubov transformation*. By a trivial Bogoliubov transformation the BD modes can be chosen to obey

$$f_{\boldsymbol{k}}^{*}(x) = f_{\boldsymbol{k}}(\bar{x}), \tag{3.42}$$

where $\bar{x}$ represents the antipodal point of x. This transformation does not mix positive and negative frequency modes and therefore defines an equivalent vacuum state [4].
But the Bogoliubov transformation does mix positive and negative frequency modes, and hence the creation and annihilation operators also become dependent on the Bogoliubov coefficients. Therefore we can express our field as

$$\phi^{(\alpha,\beta)}(x) = \int d\!\!\bar{}^{\,d-1}k \left[\hat{a}_{\boldsymbol{k}}^{(\alpha,\beta)} f_{\boldsymbol{k}}^{(\alpha,\beta)*}(x) + \hat{a}_{\boldsymbol{k}}^{(\alpha,\beta)\dagger} f_{\boldsymbol{k}}^{(\alpha,\beta)}(x)\right], \tag{3.43}$$

where $\phi^{(\alpha,\beta)}(x) = \phi(x)$, the superscript is only a reminder in terms of which modes we are expanding. Since we define a Fock space vacuum through the annihilation operator,

$$\hat{a}_{\boldsymbol{k}}^{(\alpha,\beta)} \big|\, 0^{(\alpha,\beta)} \big\rangle = 0, \tag{3.44}$$

we also obtain a full class of two parameter (α,β)-vacua.
We can relate creation and annihilation operators to each other by substituting the new modes, eq. 3.41 into eq. 3.43,

$$a_{\boldsymbol{k}} = \cosh\alpha\, a_{\boldsymbol{k}}^{(\alpha,\beta)} + \sinh\alpha\, e^{i\beta}\, a_{\boldsymbol{k}}^{(\alpha,\beta)\dagger} \tag{3.45}$$

and by inverting we find

$$a_{\boldsymbol{k}}^{(\alpha,\beta)} = \cosh\alpha\, a_{\boldsymbol{k}} - \sinh\alpha\, e^{i\beta}\, a_{\boldsymbol{k}}^{\dagger}. \tag{3.46}$$

Alternatively, we can express this Bogoliubov transformation through a unitary operator [16, 17]

$$\begin{aligned} a_{\boldsymbol{k}}^{(\alpha,\beta)} &= B^{(\alpha,\beta)}\, a_{\boldsymbol{k}}\, B^{(\alpha,\beta)\dagger} \quad \text{with} \\ B^{(\alpha,\beta)} &:= \exp\left(\frac{1}{2}\int d^{d-1}k\, \alpha \left(e^{i\beta} a_{\boldsymbol{k}}^{\dagger 2} - e^{-i\beta} a_{\boldsymbol{k}}^{2}\right)\right), \end{aligned} \tag{3.47}$$

a proof of which we give in app. B. Therefore, even though the transformation mixed positive and negative frequency modes, any physical observables are left invariant by unitary transformations [17]. Hence we can also write the new vacuum in terms of the BD one as squeezed states [18]

$$\big|\, 0^{(\alpha,\beta)} \big\rangle = B^{(\alpha,\beta)} \big|\, 0 \big\rangle. \tag{3.48}$$

This transformation leads to states invariant under the proper (time preserving) de Sitter group $SO(1,d)$ and are known as Mottola-Allen vacua [4, 16]. For clarity we will refer the this class as (α,β)-vacua. Note any α vacuum will contain a particle spectrum with respect to the BD vacuum or any other $\alpha' \neq \alpha$ vacuum as we will show shortly. We can express the unitary operator representing the Bogoliubov transformation as [19, 20]

$$B^{(\alpha,\beta)} = e^{\frac{1}{2} e^{i\beta} \tanh\alpha \int d^{d-1}k\, a_{\boldsymbol{k}}^{\dagger 2}} \left(\frac{1}{\cosh\alpha}\right)^{\int d^{d-1}k\, a_{\boldsymbol{k}}^{\dagger} a_{\boldsymbol{k}} + \frac{1}{2}} e^{-\frac{1}{2} e^{-i\beta} \tanh\alpha \int d^{d-1}k\, a_{\boldsymbol{k}}^{2}}. \tag{3.49}$$

Therefore we can express the (α,β)-vacuum through the BD vacuum as

$$\left|\, 0^{(\alpha,\beta)} \right\rangle = \frac{1}{\sqrt{\cosh\alpha}} \exp\left(\frac{1}{2}\tanh\alpha\, e^{i\beta} \int d^{d-1}k\, a_{\boldsymbol{k}}^{\dagger 2}\right) |\, 0\rangle, \tag{3.50}$$

which includes only even occupation number states. From this we can easily see that the overlap of any transformed vacuum with the BD vacuum is

$$\left\langle 0 \,|\, 0^{(\alpha,\beta)} \right\rangle = \frac{1}{\sqrt{\cosh\alpha}}. \tag{3.51}$$

We can even compute the overlap of two different vacua as [20]

$$\left\langle 0^{(\alpha,\beta)} \,|\, 0^{(\alpha',\beta')} \right\rangle = \frac{1}{\sqrt{\cosh\alpha' \, \cosh\alpha \left(1 - \tanh\alpha' \, \tanh\alpha\, e^{i(\beta'-\beta)}\right)}}, \tag{3.52}$$

from which we can also reproduce the overlap with the BD vacuum by setting $\alpha = 0$. Hence we see that different (α,β)-vacua are not orthogonal.
Any (α,β)-vacua will contain particles with respect to the BD vacuum. We can easily compute the number of particles by calculating the expectation value of the (α,β)-number operator for each mode, $n_{\boldsymbol{k}}^{(\alpha,\beta)} = a_{\boldsymbol{k}}^{(\alpha,\beta)\dagger} a_{\boldsymbol{k}}^{(\alpha,\beta)}$ in the BD vacuum,

$$\begin{aligned}
&\left\langle 0 \left| n_{\boldsymbol{k}}^{(\alpha,\beta)} \right| 0 \right\rangle = \left\langle 0^{(\alpha,\beta)} \,|\, n_{\boldsymbol{k}} \,|\, 0^{(\alpha,\beta)} \right\rangle \\
&= \left\langle 0 \left| B^{(\alpha,\beta)}\, a_{\boldsymbol{k}}^{\dagger}\, B^{(\alpha,\beta)\dagger} B^{(\alpha,\beta)}\, a_{\boldsymbol{k}}\, B^{(\alpha,\beta)\dagger} \right| 0 \right\rangle \\
&= \left\langle 0 \left| \left(\cosh\alpha\, a_{\boldsymbol{k}}^{\dagger} - \sinh\alpha\, e^{-i\beta}\, a_{\boldsymbol{k}}\right) \left(\cosh\alpha\, a_{\boldsymbol{k}} - \sinh\alpha\, e^{i\beta}\, a_{\boldsymbol{k}}^{\dagger}\right) \right| 0 \right\rangle \\
&= \sinh^2\alpha,
\end{aligned} \tag{3.53}$$

where $n_{\boldsymbol{k}} = n_{\boldsymbol{k}}^{(0,0)}$ is the BD number operator. Hence we see that for different α we can get a virtually infinite particle contribution with respect to any other (α',β')-vacuum, including the BD one.

3.5 The euclidean two point function

The two point function encodes enough information to construct any other Green function from it (see app. C) and is therefore a very useful quantity. There are two ways to obtain a mathematical expression for the two point function. The most direct is the substitution of the mode expansion 3.39 into the definition,

$$G_+(x,y) = \langle 0 \,|\, \phi(x)\phi(y) \,|\, 0\rangle\,. \tag{3.54}$$

This approach is quite involved mathematically, but has the advantage, that the $i\epsilon$-prescription is fixed automatically. This approach is discussed in detail in app. D and the final result for the BD two point function is

$$G_+(Z_\epsilon) = \frac{H^{d-2}}{(4\pi)^{d/2}} \frac{\Gamma(N_-)\,\Gamma(N_+)}{\Gamma\left(\frac{d}{2}\right)}\, {}_2F_1\left(N_-, N_+; \frac{d}{2}; \frac{1+Z_\epsilon}{2}\right), \tag{3.55}$$

where $n = \sqrt{\left(\frac{d-1}{2}\right)^2 - \frac{m^2}{H^2}}$ and $N_\pm \coloneqq \frac{d-1}{2} \pm n$. The $i\epsilon$-prescription is

$$Z_\epsilon = Z_\epsilon(x, y) \coloneqq Z(x, y) - i\epsilon \,\mathrm{sgn}(x, y), \tag{3.56}$$

where $\mathrm{sgn}(x, y) \coloneqq \Theta(\eta_x - \eta_y) - \Theta(\eta_y - \eta_x)$. The function $Z_\epsilon(x, y)$ carries the extra information of the time ordering of x and y, but is only invariant under the time preserving part of the de Sitter group $SO(1, d)$.

The second approach is much more elegant. We can also find the two point function $G_+(x, y)$ using the assumption of de Sitter invariance as an Ansatz. Looking at the result we obtained the long way, eq. 3.55, we see that the two point function is only a function of the geodesic distance (or equivalently, of Z), up to the $i\epsilon$-prescription. By assuming invariance under the full de Sitter group, we can arrive at the result for the two point function much quicker and also work in a coordinate independent framework. The disadvantage is, that we have to fix the $i\epsilon$-prescription subsequently by hand. We will follow Allen [4] quite closely here. As we have seen in app. C, for the case of a free, massive scalar field, generic Green functions satisfy

$$(\Box + m^2)\mathcal{G}(x, y) = 0. \tag{3.57}$$

As we require de Sitter invariance of our vacuum, our two point function can not depend on x and y directly, but must be a function of the de Sitter invariant geodesic distance, $D(x, y)$, or equivalently the function $Z(x, y)$. Therefore we must express the covariant d'Alembert operator in terms of $Z(x, y)$,

$$\Box\mathcal{G}(Z) = \frac{\mathrm{d}^2\mathcal{G}}{\mathrm{d}Z^2} g^{\mu\nu}\partial_\mu Z \partial_\nu Z + \frac{\mathrm{d}\,\mathcal{G}}{\mathrm{d}Z}\Box Z. \tag{3.58}$$

Furthermore, using the results from sec. 2 one easily finds that

$$g^{\mu\nu}\partial_\mu Z \partial_\nu Z = -H^2(1 - Z^2) \quad \text{and} \quad \Box Z = dH^2 Z. \tag{3.59}$$

Using these results, we can rewrite eq. 3.57 and find a differential equation for the two point function,

$$\left[(Z^2 - 1)\frac{\mathrm{d}^2}{\mathrm{d}Z^2} + dZ\frac{\mathrm{d}}{\mathrm{d}Z} + \frac{m^2}{H^2}\right]\mathcal{G}(Z) = 0, \tag{3.60}$$

which due to its symmetry in $Z \to -Z$ admits a solution $\mathcal{G}(-Z)$, given $\mathcal{G}(Z)$ is a solution. Substituting $Z = 2z - 1$, we obtain the differential equation for the hypergeometric function [8],

$$\left[z(1 - z)\frac{\mathrm{d}^2}{\mathrm{d}z^2} + d\left(\frac{1}{2} - z\right)\frac{\mathrm{d}}{\mathrm{d}Z} - \frac{m^2}{H^2}\right]\mathcal{G}(Z) = 0. \tag{3.61}$$

The general solutions to this equation are the hypergeometric functions [1],

$$\mathcal{G}(Z) = C_1\, {}_2F_1\left(N_-, N_+; \frac{d}{2}; \frac{1 + Z}{2}\right) + C_2\, {}_2F_1\left(N_-, N_+; \frac{d}{2}; \frac{1 - Z}{2}\right), \tag{3.62}$$

[1] Compare with the differential equation of the hypergeometric differential equation [39],

$$z(z - 1)\frac{\mathrm{d}^2 F}{\mathrm{d}z^2} + (c - (a + b + 1)z)\frac{\mathrm{d}\,F}{\mathrm{d}z} - abF = 0,$$

to which the hypergeometric functions ${}_2F_1(a, b; c; z)$ provide solutions. For eq. 3.61,

$$a = \frac{d - 1}{2} \pm n, \qquad b = \frac{d - 1}{2} \mp n, \qquad c = \frac{d}{2}, \qquad n = \sqrt{\left(\frac{d - 1}{2}\right)^2 - \frac{m^2}{H^2}}.$$

It does not matter which signs we take for a, b, as ${}_2F_1(a, b; c; z) = {}_2F_1(b, a; c; z)$ [4].

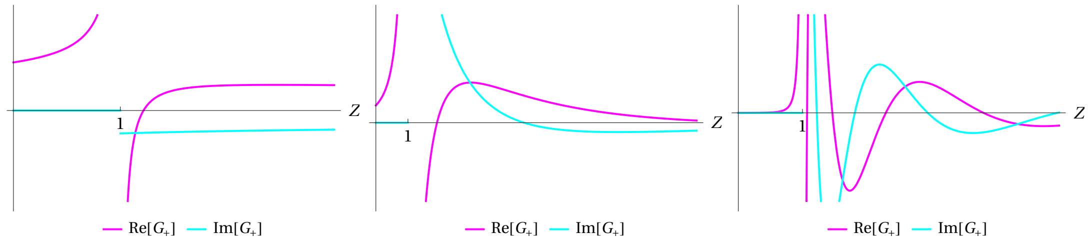


Figure 3.1: The real and imaginary parts of the BD Green function, the first term in eq. 3.62, are shown here for different values of m^2/H^2: left $m^2/H^2 = 1/4$ such that n is purely real, center $m^2/H^2 = 9/4$ such that $n = 0$, right $m^2/H^2 = 24/4$ such that n is purely imaginary. One clearly sees the complicated behaviour at the coincidence point $Z = 1$. Additionally, we can observe decaying behaviour in the left hand figure, where n is purely real and damped oscillatory behaviour for the right hand figure, where n is purely imaginary.

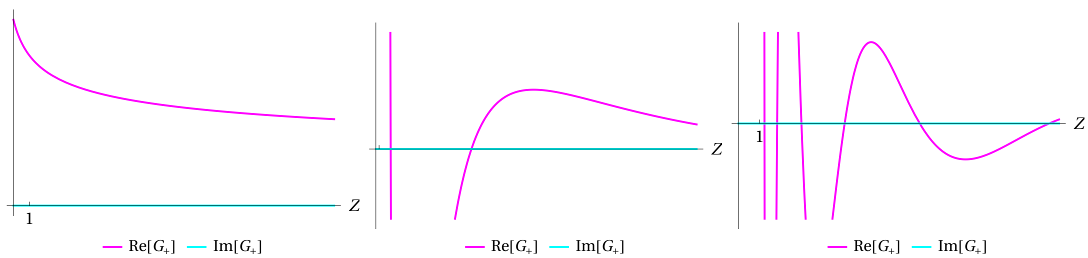


Figure 3.2: The real and imaginary parts of the second term in eq. 3.62, which corresponds to the BD two point function with $Z \to -Z$ are shown here for different values of m^2/H^2: left $m^2/H^2 = 1/4$ such that n is purely real, centre $m^2/H^2 = 9/4$ such that $n = 0$, right $m^2/H^2 = 24/4$ such that n is purely imaginary. The first thing that one clearly observes, is that $\mathrm{Im}\,(G_+) = 0$ for any value of n. Additionally, the discontinuous behaviour at the coincidence point $Z = 1$, has disappeared. The reason for this becomes clear by looking at the defining series expansion of the hypergeometric function, for which ${}_2F_1(a, b, c; z) \sim \sum_{s=0}^{\infty} \frac{z^s}{s!}$ [39]. For this function we have $z = 0$ at the coincidence limit. Additionally, we can again observe damped oscillatory behaviour for the right hand figure, where n is purely imaginary.

where $C_{1/2}$ are some complex coefficients. The hypergeometric function ${}_2F_1(a, b, c; z)$ has poles at $z = 0, 1, \infty$ and a branch cut from $z = 1$ to $z = \infty$ along the real axis [39]. Therefore, the first term in eq. 3.62 has a pole at null separation of x and y and the second term has a pole at null separation of x and $\bar{y}$. The behaviour of the first term (including normalisation), which corresponds to the BD solution, is shown in fig. 3.1. The second term (including normalisation) is investigated in fig. 3.2.

Since the results we obtained for the Green function depends only on the geodesic distance, this expression holds equally well in global de Sitter spacetime, where the coordinates cover the complete manifold. In the global patch, the BD vacuum is referred to as the *euclidean vacuum*, due to its connection to the d-sphere for analytically continued time. The Euclidean vacuum will be denoted by $\big|0\big\rangle$, equivalently to the BD vacuum. Consequently, the Green function in the global patch is usually referred to as the *euclidean Green function.* One defines the euclidean Green function as the solution, which has only one singularity when x is on the light cone of y [4]. The normalisation coefficient C_1 can be determined by requiring the same singularity behaviour as in Minkowski space in the UV-limit, $Z \to 1$. For the euclidean Green function we therefore obtain [8]

$$\mathcal{G}^E(Z) = \frac{H^{d-2}}{(4\pi)^{d/2}} \frac{\Gamma(N_-)\Gamma(N_+)}{\Gamma\left(\frac{d}{2}\right)} {}_2F_1\left(N_-, N_+; \frac{d}{2}; \frac{1+Z}{2}\right). \tag{3.63}$$

From fig. 3.1 we see the different behaviour for different values of m^2/H^2. The case where n is real and therefore $\left(\frac{d-1}{2}\right)^2 > \frac{m^2}{H^2}$ is known as the *complementary series*. Here we observe a decaying behaviour of the Green function. The second case, where n is imaginary and therefore $\left(\frac{d-1}{2}\right)^2 < \frac{m^2}{H^2}$ is known as the *principle series*. Here we observe damped oscillatory behaviour of the Green function.
Now we can use this result to compute the two point function and the other related Green functions by introducing the correct pole structure via the $i\epsilon$-prescription. For the two point function we use the same $i\epsilon$-prescription as above

$$\begin{aligned} G_+(x,y) &= G_+(Z_\epsilon) = \mathcal{G}^E(Z - i\epsilon\, \mathrm{sgn}(x,y)) \\ &= \frac{H^{d-2}}{(4\pi)^{d/2}} \frac{\Gamma(N_+)\Gamma(N_-)}{\Gamma\left(\frac{d}{2}\right)} {}_2F_1\left(N_-, N_+; \frac{d}{2}; \frac{1+Z_\epsilon}{2}\right), \end{aligned} \tag{3.64}$$

which then completely agrees with our previous result in eq. 3.55. Since we have introduced explicit time ordering through the $i\epsilon$-prescription, the two point function $G_+(Z_\epsilon)$ is generally only invariant under the time preserving subgroup $SO(1,d)$ [18].
As a further example, the euclidean Feynman Green function can easily be constructed from its definition in eq. C.14 and can be expressed as

$$G_F(x,y) = \mathcal{G}^E(Z - i\epsilon). \tag{3.65}$$

One very interesting case to consider is the conformally coupled, massless case where the effective mass (including the ξR coupling term in the action) is

$$m^2 - \xi R = \frac{d(d-2)}{4} H^2 \quad \text{and} \quad n = \frac{1}{2}. \tag{3.66}$$

In this case the euclidean two point function becomes

$$G_+(Z_\epsilon) = \frac{H^{d-2}}{(4\pi)^{d/2}} \Gamma\left(\frac{d}{2} - 1\right) \left(\frac{2}{1 - Z_\epsilon}\right)^{\frac{d}{2}-1}. \tag{3.67}$$

For $d = 4$ this further simplifies to

$$G_+(Z_\epsilon) = \frac{H^2}{8\pi^2} \frac{1}{1 - Z_\epsilon}. \tag{3.68}$$

Here the pole structure is directly visible, which allows us to easily compare to the flat space case. Now we can clearly see the pole position at $Z = 1$, where x and y have lightlike separation.

3.6 The (α, β)-two point function

Having constructed the euclidean two point function we can ask again, what if we had chosen any other of the class of (α, β)-modes? Since two potentially different (α, β)-modes can enter in the two point function, this gives us a four parameter freedom.
We can use eq. 3.45 and eq. 3.46 to relate the two different operators $a_{\boldsymbol{k}}^{(\alpha,\beta)}$ and $a_{\boldsymbol{k}}^{(\alpha',\beta')}$. One finds,

$$\begin{aligned} a_{\boldsymbol{k}}^{(\alpha,\beta)} =& (\cosh\alpha\cosh\alpha' - \sinh\alpha\sinh\alpha' e^{i(\beta-\beta')})\, a_{\boldsymbol{k}}^{(\alpha',\beta')} \\ &+ (\cosh\alpha\sinh\alpha' e^{i\beta'} - \sinh\alpha\cosh\alpha' e^{i\beta})\, a_{\boldsymbol{k}}^{(\alpha',\beta'),\dagger} \\ =:& \gamma a_{\boldsymbol{k}}^{(\alpha',\beta')} + \delta a_{\boldsymbol{k}}^{(\alpha',\beta')\dagger}. \end{aligned} \tag{3.69}$$

Now we want to calculate the general two point function for two different vacua [7],

$$G_+^{(\alpha,\beta),(\alpha',\beta')}(x,y) = \frac{\left\langle 0^{(\alpha,\beta)} \mid \phi(x)\phi(y) \mid 0^{(\alpha',\beta')} \right\rangle}{\left\langle 0^{(\alpha,\beta)} \mid 0^{(\alpha',\beta')} \right\rangle}. \tag{3.70}$$

Now we substitute the mode expansion from eq. 3.43, in terms of the modes, corresponding to the respective vacua

$$\begin{aligned} &G_+^{(\alpha,\beta),(\alpha',\beta')}(x,y) = \\ &= \frac{1}{2}\int đ^{d-1}k\, đ^{d-1}q \frac{\left\langle 0^{(\alpha,\beta)} \middle| a_{\boldsymbol{k}}^{(\alpha,\beta)} a_{\boldsymbol{q}}^{(\alpha',\beta')\dagger} \middle| 0^{(\alpha',\beta')} \right\rangle}{\left\langle 0^{(\alpha,\beta)} \mid 0^{(\alpha',\beta')} \right\rangle} f_{\boldsymbol{k}}^{(\alpha,\beta)*}(x) f_{\boldsymbol{q}}^{(\alpha',\beta')}(y). \end{aligned} \tag{3.71}$$

An expression for the matrix element can be found using the conjugate of eq. 3.69 [7],

$$\frac{\left\langle 0^{(\alpha,\beta)} \middle| a_{\boldsymbol{k}}^{(\alpha,\beta)} a_{\boldsymbol{q}}^{(\alpha',\beta'),\dagger} \middle| 0^{(\alpha',\beta')} \right\rangle}{\left\langle 0^{(\alpha,\beta)} \mid 0^{(\alpha',\beta')} \right\rangle} = \frac{1}{\gamma^*}\delta^{d-1}(\boldsymbol{k}-\boldsymbol{q}). \tag{3.72}$$

Using this and the Bogoliubov transformation to the euclidean modes in eq. 3.41, we can write the general two point function in terms of a superposition of euclidean two point functions, eq. 3.55,

$$\begin{aligned} G_+^{(\alpha,\beta),(\alpha',\beta')}(x,y) = \frac{1}{\gamma^*}[&\cosh\alpha\cosh\alpha' G_+(x,y) + \sinh\alpha\sinh\alpha' e^{-i(\beta-\beta')} G_+(\bar{x},\bar{y}) \\ &+ \cosh\alpha\sinh\alpha' e^{i\beta'} G_+(x,\bar{y}) + \sinh\alpha\cosh\alpha' e^{-i\beta} G_+(\bar{x},y)], \\ \gamma^* =& \cosh\alpha\cosh\alpha' - \sinh\alpha\sinh\alpha' e^{-i(\beta-\beta')}, \end{aligned} \tag{3.73}$$

where we made use of the property in eq. 3.42. It is easily seen that from $G_+^{(\alpha,\beta),(\alpha',\beta')}$ one can obtain the euclidean two point function G_+, if either α or α' is set to zero. Using the property $Z(\bar{x}, y) = -Z(x, y)$ and $Z(x, \bar{y}) = -Z(x, y)$, we can rewrite the above expression in terms of Z

$$\begin{aligned} G_+^{(\alpha,\beta),(\alpha',\beta')}(Z_\epsilon) = \frac{1}{\gamma^*}[&\cosh\alpha\cosh\alpha' G_+(Z_\epsilon) + \sinh\alpha\sinh\alpha' e^{-i(\beta-\beta')} G_+(Z_{-\epsilon}) \\ &+ \cosh\alpha\sinh\alpha' e^{i\beta'} G_+(-Z_{\bar{\epsilon}}) + \sinh\alpha\cosh\alpha' e^{-i\beta} G_+(-Z_{-\bar{\epsilon}})], \\ \gamma^* =& \cosh\alpha\cosh\alpha' - \sinh\alpha\sinh\alpha' e^{-i(\beta-\beta')}. \end{aligned} \tag{3.74}$$

Here the notation is the following, $\{\pm\}Z_{\pm\bar{\epsilon}} = \{\pm\}(Z \mp i\epsilon\,\mathrm{sgn}(\bar{x},y))$, where

$$\mathrm{sgn}(\bar{x},y) = -\,\mathrm{sgn}(x,\bar{y}) = \Theta(-\eta_x - \eta_y) - \Theta(\eta_x + \eta_y), \tag{3.75}$$

as under antipodal transformations $\eta_x \to -\eta_x$. Note also that $\mathrm{sgn}(\bar{x},y)$ is symmetric under exchange of η_x and η_y.
Usually we take expectation values with respect to the same vacuum. So let us set $\alpha' = \alpha$ and $\beta' = \beta$ and rename $G_+^{(\alpha,\beta),(\alpha,\beta)} =: G_+^{(\alpha,\beta)}$. From eq. 3.73 and eq. 3.74 we find that $\gamma^* = 1$ and that

$$\begin{aligned} G_+^{(\alpha,\beta)}(x,y) =& \cosh^2\alpha\, G_+(x,y) + \sinh^2\alpha\, G_+(\bar{x},\bar{y}) \\ &+ \cosh\alpha\, \sinh\alpha\, \left(e^{i\beta} G_+(x,\bar{y}) + e^{-i\beta} G_+(\bar{x},y)\right) \\ =& \cosh^2\alpha\, G_+(Z_\epsilon) + \sinh^2\alpha\, G_+(Z_{-\epsilon}) \\ &+ \cosh\alpha \sinh\alpha\, \left(e^{i\beta} G_+(-Z_{\bar{\epsilon}}) + e^{-i\beta} G_+(-Z_{-\bar{\epsilon}})\right). \end{aligned} \tag{3.76}$$

Let us now check what happens under time reversal. Time reversal and the antipodal transformation are both elements of $O_T(1,d)$, hence we can just preform an antipodal transformation under which,

$$\begin{aligned} G_+^{(\alpha,\beta)}(T\{x\},T\{y\}) =& G_+^{(\alpha,\beta)}(\bar{x},\bar{y}) \\ =& \cosh^2\alpha\, G_+(\bar{x},\bar{y}) + \sinh^2\alpha\, G_+(x,y) \\ &+ \cosh\alpha\, \sinh\alpha\, \left(e^{i\beta} G_+(\bar{x},y) + e^{-i\beta} G_+(x,\bar{y})\right) \\ =& \cosh^2\alpha\, G_+(y,x) + \sinh^2\alpha\, G_+(\bar{y},\bar{x}) \\ &+ \cosh\alpha\, \sinh\alpha\, \left(e^{i\beta} G_+(\bar{y},x) + e^{-i\beta} G_+(y,\bar{x})\right), \end{aligned} \tag{3.77}$$

where we denote time reversal by $T\{.\}$ and have made use of the property

$$G_+(\bar{x},\bar{y}) = G_+(y,x), \quad \text{or} \quad T\{Z_\epsilon\} = Z_{-\epsilon}. \tag{3.78}$$

Therefore we see, that the two point function is only time reversal invariant in its general form if we set $\beta = 0$, but its arguments are reversed. Setting $\beta = 0$ we find

$$G_+^{(\alpha,0)}(T\{x\},T\{y\}) = G_+^{(\alpha,0)}(T\{Z_\epsilon\}) = G_+^{(\alpha,0)}(Z_{-\epsilon}) = G_+^{(\alpha,0)}(y,x). \tag{3.79}$$

We have seen, that a frequency independent Bogoliubov transformation leaving orthogonality of the modes invariant leads to de Sitter invariance under the time preserving group $SO(1,d)$. We can fix the value of α (and β) by matching out solution onto the flat spacetime theory in the UV limit, leading to the euclidean or BD vacuum, as we did above. This is physically motivated and fixes the value of α to zero, but we still have to regard the other (α,β)-vacua as physically equivalent.

3.7 Green functions and de Sitter invariance

Now the question arises, if the two point functions in the new family of (α,β)-vacuum states are still de Sitter invariant. Consider the mode expansion of the commutator and anticommutator two point functions after the transformation

$$\begin{aligned} iG_{(\alpha,\beta)}(x,y) &= \left\langle 0^{(\alpha,\beta)} \,\middle|\, \left[\phi^{(\alpha,\beta)}(x), \phi^{(\alpha,\beta)}(y)\right] \,\middle|\, 0^{(\alpha,\beta)} \right\rangle \\ &= \int đ^{d-1}k \left[f_{\boldsymbol{k}}^{(\alpha,\beta)*}(x)\; f_{\boldsymbol{k}}^{(\alpha,\beta)}(y) - f_{\boldsymbol{k}}^{(\alpha,\beta)*}(y)\; f_{\boldsymbol{k}}^{(\alpha,\beta)}(x) \right], \\ G^{(1)}_{(\alpha,\beta)}(x,y) &= \left\langle 0^{(\alpha,\beta)} \,\middle|\, \left\{\phi^{(\alpha,\beta)}(x), \phi^{(\alpha,\beta)}(y)\right\} \,\middle|\, 0^{(\alpha,\beta)} \right\rangle \\ &= \int đ^{d-1}k \left[f_{\boldsymbol{k}}^{(\alpha,\beta)*}(x)\; f_{\boldsymbol{k}}^{(\alpha,\beta)}(y) + f_{\boldsymbol{k}}^{(\alpha,\beta)*}(y)\; f_{\boldsymbol{k}}^{(\alpha,\beta)}(x) \right]. \end{aligned} \tag{3.80}$$

Now substituting eq. 3.41 for the mode functions $f_{\boldsymbol{k}}^{(\alpha,\beta)}$ we obtain

$$\begin{aligned} iG_{(\alpha,\beta)}(x,y) =& (\cosh^2\alpha - \sinh^2\alpha) \int đ^{d-1}k \,[f^*_{\boldsymbol{k}}(x) f_{\boldsymbol{k}}(y) - f_{\boldsymbol{k}}(x) f^*_{\boldsymbol{k}}(y)] \\ =& \int đ^{d-1}k \,[f^*_{\boldsymbol{k}}(x) f_{\boldsymbol{k}}(y) - f_{\boldsymbol{k}}(x) f^*_{\boldsymbol{k}}(y)] \end{aligned} \tag{3.81}$$

for the commutator and

$$\begin{aligned} G^{(1)}_{(\alpha,\beta)}(x,y) =& \cosh(2\alpha) \int đ^{d-1}k \,[f^*_{\boldsymbol{k}}(x) f_{\boldsymbol{k}}(y) + f_{\boldsymbol{k}}(x) f^*_{\boldsymbol{k}}(y)] \\ &+ \sinh(2\alpha)\cos\beta \int đ^{d-1}k \,[f^*_{\boldsymbol{k}}(x) f^*_{\boldsymbol{k}}(y) + f_{\boldsymbol{k}}(x) f_{\boldsymbol{k}}(y)] \\ &+ i\sinh(2\alpha)\sin\beta \int đ^{d-1}k \,[f_{\boldsymbol{k}}(x) f_{\boldsymbol{k}}(y) - f^*_{\boldsymbol{k}}(x) f^*_{\boldsymbol{k}}(y)] \end{aligned} \tag{3.82}$$

for the anticommutator. Using the property of the euclidean mode functions given by eq. 3.42, we can write the Bogoliubov transformed Green function, $G^{(1)}_{(\alpha,\beta)}(x,y)$ in terms of the euclidean two point functions

$$\begin{aligned} iG(x,y) = iG_{(0,0)}(x,y) = \langle 0 \,|\, [\phi(x), \phi(y)] \,|\, 0\rangle = G_+(x,y) - G_+(y,x), \\ G^{(1)}(x,y) = G^{(1)}_{(0,0)}(x,y) = \langle 0 \,|\, \{\phi(x), \phi(y)\} \,|\, 0\rangle = G_+(x,y) + G_+(y,x). \end{aligned} \tag{3.83}$$

$G^{(1)}(x,y)$ and similarly $G(x,y)$ can be expressed through $G_+(x,y)$ and $G_+(y,x)$. They therefore inherit invariance under the proper de Sitter group from the two point function. Furthermore, under time reversal,

$$\begin{aligned} iG(T\{x\}, T\{y\}) = -iG(x,y), \\ G^{(1)}(T\{x\}, T\{y\}) = G^{(1)}(x,y), \end{aligned} \tag{3.84}$$

the commutator changes sign and the anticommutator remains completely invariant. The euclidean anticommutator two point function is therefore invariant under the full de Sitter group [4]. The commutator two point function

$$iG_{(\alpha,\beta)}(x,y) = iG(x,y), \tag{3.85}$$

is found to be independent of the vacuum chosen. Furthermore due to the general relation to advanced and retarded Green functions,

$$G_A(x,y) = \theta(\eta_y - \eta_x) G(x,y) \quad \text{and} \quad G_R(x,y) = -\theta(\eta_x - \eta_y) G(x,y), \tag{3.86}$$

are also independent of the choice of vacuum. For the symmetric two point function we find [2]

$$G^{(1)}_{(\alpha,\beta)}(x,y) = \cosh(2\alpha) G^{(1)}(x,y) + \sinh(2\alpha)\left[\cos\beta\; G^{(1)}(\bar{x},y) - \sin\beta\; G(\bar{x},y)\right]. \tag{3.87}$$

Let us now consider what happens under time reversal. From our discussion in the previous section leading to eq. 3.79, we already know that $\beta \neq 0$ leads to a violation of time reversal invariance. Also from eq. 3.79 we know that under time reversal the arguments of the euclidean two point function simply flip. Therefore the sign of the commutator two point function changes under time reversal and we have

$$\begin{aligned} iG_{(\alpha,\beta)}(T\{x\},T\{y\}) &= -iG_{(\alpha,\beta)}(x,y), \\ G^{(1)}_{(\alpha,\beta)}(T\{x\},T\{y\}) &= G^{(1)}_{(\alpha,-\beta)}(x,y), \end{aligned} \tag{3.88}$$

which tells us that if we set $\beta = 0$, the anticommutator two point function remains invariant under the full de Sitter group.

Recalling the relations of the various Green functions described in app. C, the Feynman Green's function can be expressed as

$$iG_F^{(\alpha,\beta)}(x,y) = \frac{1}{2} G^{(1)}_{(\alpha,\beta)}(x,y) + \frac{1}{2}\,\mathrm{sgn}(x,y)\, iG_{(\alpha,\beta)}(x,y). \tag{3.89}$$

Substituting the above results, the Feynman Green's function becomes

$$iG_F^{(\alpha,\beta)}(x,y) = iG_F(x,y) + \frac{1}{2}\left[G^{(1)}_{(\alpha,\beta)}(x,y) - G^{(1)}(x,y)\right], \tag{3.90}$$

where an additional term appears with respect to the euclidean case. Under time reversal,

$$iG_F^{(\alpha,\beta)}(T\{x\},T\{y\}) = iG_F(x,y) + \frac{1}{2}\left[G^{(1)}_{(\alpha,-\beta)}(x,y) - G^{(1)}(x,y)\right], \tag{3.91}$$

we see that in addition to the anticommutator, also the Feynman Green function is time reversal invariant upon setting $\beta = 0$ and therefore remains invariant under $O(1,d)$.

We can now do a simple consistency check to confirm what we have discovered in the previous section. In eq. 3.85 we have found that the commutator Green function is independent of the chosen vacuum. This can be easily checked by constructing the commutator from eq. 3.76,

$$iG^{(\alpha,\beta),(\alpha,\beta)}(x,y) = G_+^{(\alpha,\beta),(\alpha,\beta)}(x,y) - G_+^{(\alpha,\beta),(\alpha,\beta)}(y,x). \tag{3.92}$$

Now we can use the above properties to evaluate each term separately,

$$\begin{aligned} G_+(x,y) - G_+(y,x) &= iG(x,y), \\ G_+(\bar{x},y) - G_+(\bar{y},x) &= G_+(\bar{x},y) - G_+(\bar{x},y) = 0, \\ G_+(x,\bar{y}) - G_+(y,\bar{x}) &= G_+(x,\bar{y}) - G_+(x,\bar{y}) = 0, \\ G_+(\bar{x},\bar{y}) - G_+(\bar{y},\bar{x}) &= -\left[G_+(x,y) - G_+(x,y)\right] = -iG(x,y). \end{aligned} \tag{3.93}$$

[2] Allen [4] argues that, $G(x,y)$ can not be a function of $Z(x,y)$ as the commutator two point function is antisymmetric in its arguments, $G(x,y) = -G(y,x)$, but $Z(x,y)$ is symmetric, $Z(x,y) = Z(y,x)$. Therefore he concludes, that we must set $\beta = 0$ to maintain de Sitter invariance. If we neglect the $i\epsilon$-prescription, this argument holds and $G(x,y) = 0$ in fact. We only get a non-vanishing result for timelike separated points, where $Z > 1$, if we take the $i\epsilon$-prescription into account. But this result is then invariant under the proper de Sitter group, similarly to the two point function.

Hence we see that the second and third terms in the difference of two point functions vanish and we are left with

$$iG^{(\alpha,\beta),(\alpha,\beta)}(x,y) = iG(x,y), \tag{3.94}$$

as anticipated and the commutator two point function is indeed independent of the vacuum choice.

In summary, starting from the euclidean vacuum state, we have identified a two real parameter family of symmetric and antisymmetric two point functions which are invariant under the time direction preserving, connected de Sitter group $SO(1,d)$. Additionally, for $\beta = 0$ our resulting anticommutator and Feynman two point functions are also time reversal invariant and hence the only two Green functions which are invariant under the full de Sitter group $O(1,d)$ [4, 18]. The $(\alpha,0)$-Bogoliubov transformations defines a one parameter set known as *α-vacua*.

Chapter 4

The energy-momentum tensor of a quantised scalar field

In this section our main goal will be to derive a regularised closed form expression for the quantum expectation value of the stress-energy or energy-momentum tensor (EMT) of a scalar field in the expanding Poincaré patch of de Sitter space. The motivation for this is to investigate how the quantum nature of our field can affect the geometry of the spacetime. Recall that de Sitter space is a solution to the Einstein equation for a universe dominated by the cosmological constant only. So any matter contribution will necessarily affect the spacetime unless it is negligibly small.

We will start by introducing the formalism of the effective action and the main ideas of semi-classical gravity in sec. 4.1 [1, 22, 23] and sec. 4.2 [14, 15, 24], respectively. We will see that the effective action is a generally very useful quantity, but suffers from UV divergences when computed directly. To regularise the effective action we will spend some time in sec. 4.3 [14, 15, 22, 25–27] identifying these divergences and removing them via dimensional regularisation. One result that we will discover on the way is that the quantum EMT loses its traceless property in the conformally invariant case, making it fundamentally different to the classical case. This is known as conformal or trace anomaly, which is discussed in sec. 4.4 [15].

After all this work and the slight detours we can use the obtained results to derive a closed form expression for the EMT in the BD case in sec. 4.5 [10, 15, 28]. We analyse the result in sec. 4.6 in terms of the semi-classical Einstein equation. Finally, in sec. 4.7 we generalize the result for the BD EMT to the case of general (α, β)-vacua.

4.1 The effective action

Consider a system with action $\mathcal{S}[\varphi, J, g_{\mu\nu}] = S[\varphi, g_{\mu\nu}] + \int d^d x \sqrt{|g|} J\varphi$, where $S[\varphi, g_{\mu\nu}]$ is some action of a scalar field $\varphi(x)$ and $J(x)$ represents an external source coupled to this field. In the path integral formulation,

$$Z[J, g_{\mu\nu}] = \int_{\varphi(x_i)=\varphi(t_i,\boldsymbol{x}_i)}^{\varphi(x_f)=\varphi(t_f,\boldsymbol{x}_f)} \mathcal{D}\varphi \, e^{i\mathcal{S}[\varphi,J,g_{\mu\nu}]} =: e^{iW[J,g_{\mu\nu}]}, \tag{4.1}$$

where t_i and t_f are some arbitrary initial an final times, respectively. In the absence of a source particle production will be absent and

$$Z[0, g_{\mu\nu}] = \langle \varphi_f \,|\, \varphi_i \rangle = \langle 0_f \,|\, 0_i \rangle, \tag{4.2}$$

representing the overlap of final and initial vacua respectively. The choice of what "initial" and "final" represent here, will determine the vacuum choice. This will be our normalisation condition.
We can decompose the scalar field into background and fluctuation fields $\varphi = \phi + \chi$, respectively [22]. Then we can do a saddle point approximation for the background field and expand the action in the fluctuation field,

$$\begin{aligned} S[\varphi, g_{\mu\nu}] =& S[\phi, g_{\mu\nu}] + \int d^d x \sqrt{|g(x)|} J(x)\phi(x) \\ &+ \frac{1}{2}\int d^d x\, d^d y\, \chi(x) \frac{\delta^2 S[\phi, g_{\mu\nu}]}{\delta\phi(x)\delta\phi(y)} \chi(y) + \mathcal{O}(\chi^3), \end{aligned} \tag{4.3}$$

where the linear term in χ vanishes as the background field satisfies the equation of motion,

$$\frac{\delta\, S[\phi, g_{\mu\nu}]}{\delta\phi(x)} + \sqrt{|g(x)|} J(x) = 0. \tag{4.4}$$

Preforming the Gaussian path integral over the second order fluctuation fields, we obtain

$$W[J, g_{\mu\nu}] = -i \ln Z[J, g_{\mu\nu}] = S[\phi, g_{\mu\nu}] + \int d^d x \sqrt{|g(x)|} J(x)\phi(x) + \frac{i}{2} \ln\det K, \tag{4.5}$$

where $K(x,y) \coloneqq \frac{\delta^2 S[\phi, g_{\mu\nu}]}{\delta\phi(x)\delta\phi(y)}$ and the background field ϕ is fixed by its equation of motion. The $S[\phi, g_{\mu\nu}]$ is just a constant term and can thus be ignored. The Lorentzian effective action $\Gamma[g_{\mu\nu}]$ equals $W[0, g_{\mu\nu}]$, so

$$\Gamma[g_{\mu\nu}] = -i \ln Z[0, g_{\mu\nu}] = \frac{i}{2} \operatorname{tr} \ln K, \tag{4.6}$$

where we have used that $\ln\det K = \operatorname{tr}\ln K$.
Generally, in the case of a static universe a stable vacuum exists and $\langle 0_f \,|\, 0_i \rangle = \langle 0 \,|\, 0 \rangle = 1$. We have seen that this is generally not true in dynamically changing spacetimes. Particle creation during the evolution of the universe is quantified by $|\langle 0_f \,|\, 0_i \rangle|^2$, where

$$|\langle 0_f \,|\, 0_i \rangle|^2 = e^{i(\Gamma[g_{\mu\nu}] - \Gamma[g_{\mu\nu}]^*)} = e^{-2\mathrm{Im}(\Gamma[g_{\mu\nu}])}. \tag{4.7}$$

Hence, the imaginary part of the effective action is a measure of this particle creation probability [22].
The equation of motion for the source J is

$$\begin{aligned} \frac{\left\langle 0_f \,\middle|\, \frac{\delta\,\mathcal{S}[\varphi, J, g_{\mu\nu}]}{\delta J(x)} \,\middle|\, 0_i \right\rangle_J}{\langle 0_f \,|\, 0_i \rangle_J} &\coloneqq \frac{1}{Z[J, g_{\mu\nu}]} \int \mathcal{D}\varphi\, \frac{\delta\,\mathcal{S}[\varphi, J, g_{\mu\nu}]}{\delta J(x)} e^{i\mathcal{S}[\varphi, J, g_{\mu\nu}]} = \\ &= \frac{1}{i} \frac{\delta}{\delta J(x)} \ln Z[J, g_{\mu\nu}] = \frac{\delta\, W[J, g_{\mu\nu}]}{\delta J(x)} = 0, \end{aligned} \tag{4.8}$$

where $\langle 0_f \,|\, 0_i \rangle_J = Z[J, g_{\mu\nu}]$, becomes a quantum expectation value, which can be computed via the effective action.

4.2 Semi-classical gravity

We want to consider the coupling of gravity to a scalar field without any attempt to quantize gravity. In other words, we want to consider the interaction of a classical background with a quantised system. To this end, we want to study the influence of a classical background on a quantum system and vice versa the possible backreaction on the classical system due to the quantum nature of the field. The goal of this section is to introduce the general formalism, starting from the path integral approach.
The total action of the combined system is

$$S[\phi, g_{\mu\nu}] = S_\phi[\phi, g_{\mu\nu}] + S_g[g_{\mu\nu}]. \tag{4.9}$$

S_ϕ is generally referred to as the matter action for a scalar field ϕ, which in this case is the action of our quantised scalar field, generalised to include curvature effects (and also possible coupling terms like $\sim R\phi^2$). S_g is the gravitational action, which describes the purely classical background. In the case of *conventional* general relativity the gravitational action is the Einstein-Hilbert action [14, 24],

$$S_g^{(0)}[g_{\mu\nu}] = -\frac{1}{16\pi}\int d^dx\sqrt{|g|}\,(R+2\Lambda)\,, \tag{4.10}$$

such that the variation with respect to the metric gives

$$\frac{\delta\, S_g^{(0)}}{\delta g^{\mu\nu}} = -\frac{\sqrt{|g|}}{16\pi}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - g_{\mu\nu}\Lambda\right), \tag{4.11}$$

which left hand side of the Einstein equation.
By treating the metric as a classical background, the path integral can be written as

$$\int \mathcal{D}\phi\, e^{iS_\phi[\phi,g_{\mu\nu}]+iS_g[g_{\mu\nu}]} = e^{i\Gamma[g_{\mu\nu}]+iS_g[g_{\mu\nu}]}, \tag{4.12}$$

which defines the effective action in this case. Hence the equation of motion for the background field becomes

$$\frac{\delta\,\Gamma[g_{\mu\nu}]}{\delta g_{\mu\nu}} + \frac{\delta\, S_g[g_{\mu\nu}]}{\delta g_{\mu\nu}} = 0 \tag{4.13}$$

For a matter action which is invariant under the generalisation to a curved background, $S_\phi[\phi, g_{\mu\nu}] = S_\phi[\phi]$, we obtain the Einstein equation for universe dominated by the cosmological constant, which was the definition of a de Sitter universe. But now we want to include coupling to a scalar field. Since we want to use pure de Sitter spacetime as background, we must assume that the quantum contribution remains so small that we can safely ignore backreaction. This is an important point and we will come back to this later.
The classical EMT is defined by

$$T_{\mu\nu} \coloneqq \frac{2}{\sqrt{|g|}}\frac{\delta\, S_\phi}{\delta g^{\mu\nu}}. \tag{4.14}$$

For the case of quantum contributions, from eq. 4.13, we must consider the variation of the effective action with respect to the metric

$$\frac{\delta\,\Gamma[g_{\mu\nu}]}{\delta g_{\mu\nu}} = \frac{\left\langle 0_f \,\middle|\, \frac{\delta S_\phi}{\delta g^{\mu\nu}} \,\middle|\, 0_i\right\rangle}{\left\langle 0_f \,\middle|\, 0_i\right\rangle} =: \frac{\sqrt{|g|}}{2}\langle T_{\mu\nu}\rangle, \tag{4.15}$$

which is the definition of the quantum EMT. We can specify the vacuum choice through the Green function chosen to construct the EMT.
Generally it is close to impossible to compute $\langle T_{\mu\nu}\rangle$ via the functional derivative of the effective action as one would be required to know $\Gamma[g_{\mu\nu}]$ for all possible $g_{\mu\nu}$ [15]. However it provides us with basic understanding of the general formalism, which we will use in the following sections to single out the UV divergences for the later regularisation of the EMT.
Putting everything together, we obtain the *semi-classical Einstein equation* for conventional general relativity

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - g_{\mu\nu}\Lambda = 8\pi \langle T_{\mu\nu}\rangle, \tag{4.16}$$

which defines *semi-classical gravity.* This incorporates the anticipated idea, given a classical background $g_{\mu\nu}$, we have introduced a quantum field ϕ which in return has a backreaction effect on the classical background [14].

4.3 The UV divergences of the effective action

Generally, when trying to construct the effective action one encounters UV divergences. To obtain a meaningful result one needs to find a method of regularising the effective action to remove these divergences. First, one needs to single out the divergent parts of the effective action, Γ_{div}. Then one can introduce counter terms in the action of the theory to absorb exactly these parts. If successful, one can safely subtract the divergent part of the effective action from the full expression. The resulting regularised effective action will be

$$\Gamma_{reg} = \Gamma - \Gamma_{div}. \tag{4.17}$$

By construction this will yield a finite result, but it will always be ambiguous up to finite regularisation terms as these would just cause a different shift in the counter terms.
To obtain this result, one can take

$$\Gamma[g_{\mu\nu}] = \frac{i}{2}\,\mathrm{tr}\ln K = -\frac{i}{2}\,\mathrm{tr}\ln \mathcal{G}, \tag{4.18}$$

where $\mathcal{G}$ is the Green function of the operator K, corresponding to the chosen boundary conditions. From the obtained expression we can then subtract Γ_{div}, which will give a finite result.
The probably easiest way to find the divergent parts of the expectation value of the EMT is via the heat kernel method. Another way is to obtain an equivalent result via a short distance expansion in terms of Riemann normal coordinates which is discussed in [15, 22]. The heat kernel method is a useful method to calculate the right hand side of eq. 4.18. Although, one must work in the euclidean signature, as we require a real, elliptic and self-adjoint differential operator. To this end it is useful to consider a general operator,

$$K = \Box + m^2 - \xi R + V(x), \tag{4.19}$$

where m is the mass of our field, the ξR term is a coupling term of our field to the background and $V(x)$ is a general potential containing possible additional terms.
Let us now analytically continue $\eta \to i\eta_E$ to the euclidean time. This then gives the analytically continued operator K_E, which is real, elliptic and self-adjoint [22]. Let $\{|\,\phi_\lambda\rangle\}$ be a complete basis of eigenstates of K_E with eigenvalues $\{\lambda\}$, such that

$$K_E|\,\phi_\lambda\rangle = \lambda|\,\phi_\lambda\rangle. \tag{4.20}$$

The *heat kernel* is defined such that it satisfies

$$(\partial_\tau + K_E)h(\tau) = 0, \tag{4.21}$$

where the parameter τ is usually referred to as the proper time. The formal solution to the above equation is

$$h(\tau) = e^{-\tau K_E}, \tag{4.22}$$

where we imposed the boundary condition $h(0) = 1$. Furthermore, we can expand this operator in terms of its eigenfunctions as

$$h(\tau) = \sum_\lambda e^{-\tau\lambda} \big|\, \phi_\lambda \big\rangle \big\langle \phi_\lambda \,\big|. \tag{4.23}$$

We can now use this to define the heat kernel in a coordinate basis,

$$h(x, y; \tau) := \langle x \,|\, h(\tau) \,|\, y\rangle = \sum_\lambda e^{-\tau\lambda} \big\langle x \,|\, \phi_\lambda \big\rangle \big\langle \phi_\lambda \,|\, y \big\rangle = \sum_\lambda e^{-\tau\lambda} \phi_\lambda(x)\phi_\lambda(y)^*, \tag{4.24}$$

with boundary condition $h(x, y; 0) = \delta^d(x - y)/\sqrt{|g|}$. The Feynman Green function is related to the heat kernel by

$$iG_F(x, y) = \int_0^\infty d\tau \, h(x, y, \tau), \tag{4.25}$$

as can easily be seen by substituting eq. 4.24,

$$iG_F(x, y) = \sum_\lambda \frac{1}{\lambda} \phi_\lambda(x)\phi_\lambda(y)^*, \tag{4.26}$$

and by applying K_E,

$$K_E iG_F(x, y) = \big\langle x \,\big|\, \left[\sum_\lambda \frac{1}{\lambda} K_E \big|\, \phi_\lambda \big\rangle \big\langle \phi_\lambda \,\big| \right] \big|\, y \big\rangle = \big\langle x \,|\, y \big\rangle = \frac{\delta^d(x - y)}{\sqrt{|g|}}, \tag{4.27}$$

where we have used that the eigenfunctions of K_E form a complete set. The obtained result is the differential equation for the inhomogeneous Green function, which is indeed satisfied by the Feynman Green function.

4.3.1 The zeta function and the effective action

As a next step it is useful to define the euclidean *zeta function* [14],

$$\zeta_E(s) = \sum_\lambda \lambda^{-s}, \tag{4.28}$$

where the Lorentzian zeta function is given by $\zeta = -i\zeta_E$. This definition is useful due to its connection to the euclidean effective action via

$$\Gamma_E[g_{\mu\nu}] = \frac{i}{2} \ln \det K_E = \frac{i}{2} \sum_\lambda \ln \lambda = \frac{i}{2} \frac{\mathrm{d}\,\zeta_E(s)}{\mathrm{d}s}\Big|_{s=0}. \tag{4.29}$$

The ζ-function is related to the heat kernel by [14]

$$\zeta_E(s) = \frac{1}{\Gamma(s)} \int_0^\infty d\tau \tau^{s-1} \operatorname{tr} h(\tau) = \frac{1}{\Gamma(s)} \int_0^\infty d\tau\, \tau^{s-1} \int d^d x \sqrt{|g_E|} h(x, x; \tau), \tag{4.30}$$

where $\sqrt{|g_E|}$ is the euclidean metric determinant. Pulling out the mass term of the differential operator, $\tilde{K}_E := K_E - m^2 = \Box_E - \xi R + V(x)$ we can write the heat kernel as

$$h(\tau) = e^{-\tau m^2} e^{-\tau \tilde{K}_E}. \tag{4.31}$$

The reason why this is useful is that it gives us control over the IR divergence.
In the coincidence limit we can expand the heat kernel for $\tau \to 0$, to find [25]

$$h(x, x; \tau) = \frac{1}{(4\pi\tau)^{d/2}} e^{-\tau m^2} \sum_n \tau^n a_n(x), \tag{4.32}$$

where the coefficients a_n are functions containing powers of curvature terms, the potential terms $\xi R + V(x)$ present in the differential operator and derivatives thereof. Alternatively, one can also derive a series expansion of the heat kernel via metric perturbation, giving a expansion in terms of curvature values. A discussion of this can be found in [14], where also the equivalence to the proper time expansion in eq. 4.32 is shown up to $n = 2$.
Substituting the above expansion into eq. 4.30 and rearranging,

$$\zeta_E(s) = \frac{1}{(4\pi)^{d/2}\Gamma(s)} \int d^d x \sqrt{|g_E|} \sum_n a_n(x) \int_0^\infty d\tau e^{-\tau m^2} \tau^{s-1} \tau^{n-d/2}, \tag{4.33}$$

we observe that the τ integral diverges at the lower bound for terms up to $n + s = d/2$. This corresponds to a UV divergence of the theory. At the upper boundary of the integral we should strictly speaking introduce an IR cutoff, as the expansion from eq. 4.32 is only valid for sufficiently small τ. Alternatively, we can use the mass as a regulator and argue that it is large enough to make the integrand vanish sufficiently quickly in the IR limit [25].
We can introduce a cutoff τ_{UV} as the lower bound of the τ integral in eq. 4.33,

$$\int_{\tau_{UV}}^\infty d\tau e^{-\tau m^2} \tau^{s-1} \tau^{n-d/2} = m^{d-2(n+s)} \Gamma\left(n + s - \frac{d}{2}, m^2 \tau_{UV}\right), \tag{4.34}$$

where $\Gamma(s, a) = \int_a^\infty dt\, t^{s-1} e^{-t}$ is the incomplete Gamma function. Using this result in 4.33 gives

$$\zeta_E(s) = \frac{1}{(4\pi)^{d/2}\Gamma(s)} \int d^d x \sqrt{|g_E|} \sum_n a_n(x) m^{d-2(n+s)} \Gamma\left(n + s - \frac{d}{2}, m^2 \tau_{UV}\right). \tag{4.35}$$

Near $s = 0$ the gamma function has the expansion

$$\frac{1}{\Gamma(s)} \approx s + \mathcal{O}(s^2). \tag{4.36}$$

Using this, we find

$$\begin{aligned} \frac{\mathrm{d}\,\zeta_E(s)}{\mathrm{d}s}\Big|_{s=0} =& \frac{1}{(4\pi)^{d/2}} \int d^d x \sqrt{|g_E|} \sum_n a_n(x) m^{d-2n} \\ & \times \left[\Gamma(n - \frac{d}{2}, m^2\tau_{UV}) + \frac{1}{\Gamma(s)} \frac{\mathrm{d}}{\mathrm{d}s} \Gamma\left(n + s - \frac{d}{2}, m^2 \tau_{UV}\right)\Big|_{s=0}\right]. \end{aligned} \tag{4.37}$$

As long as we keep the cutoff parameter, τ_{UV}, non-zero,

$$\frac{\mathrm{d}}{\mathrm{d}s}\Gamma(n+s-\frac{d}{2},m^2\tau_{UV}) = \Gamma(n+s-\frac{d}{2}-1,m^2\tau_{UV}) \tag{4.38}$$

is finite and the second term vanishes when one takes $s=0$. Additionally, we can easily rotate back from the euclidean to the Lorentzian time. One only needs to replace the euclidean metric determinant with the Lorentzian one and recall that $\zeta = -i\zeta_E$. Therefore we can finally write down an expression for our Lorentzian effective action,

$$\Gamma[g_{\mu\nu}] = \frac{1}{2(4\pi)^{d/2}} \int d^dx\sqrt{|g|}\sum_{n=0}^{\infty} a_n(x)m^{d-2n}\Gamma\left(n-\frac{d}{2},m^2\tau_{UV}\right). \tag{4.39}$$

As the Γ-function diverges in the limit $\tau_{UV}\to 0$ for $n-d/2<0$, we can conclude that the terms up to $n=d/2$ are divergent.

4.3.2 The divergent terms in the effective action

To investigate the behaviour of the singularities arising in the effective action and discuss regularisation we must specify the number of dimensions we wish to consider in the end. We will make the choice of the physically motivated $d=4$ so that the divergent terms in the effective action are $n=0,1,2$. On the other hand, in the following expressions we will keep a general d, in preparation for dimensional regularisation.
The divergent part of the effective action can thus be written as

$$\Gamma_{div}[g_{\mu\nu}] = \frac{1}{2(4\pi)^{d/2}}\left(\frac{m}{\mu}\right)^{d-4} \int d^dx\sqrt{|g|}\sum_{n=0}^{2} a_n(x)m^{4-2n}\Gamma\left(n-\frac{d}{2},m^2\tau_{UV}\right), \tag{4.40}$$

where we have introduced a mass scale μ to keep the effective action dimensionless [15, 22, 25]. The coefficients a_n can be found for example in [25]. For the case of our differential operator $K-m^2 = \Box - \xi R + V(x)$,

$$\begin{aligned}
a_0 =& 1,\\
a_1 =& -\left(\frac{1}{6}-\xi\right)R - V,\\
a_2 =& \frac{1}{180}\left(R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} - R_{\alpha\beta}R^{\alpha\beta}\right) + \frac{1}{2}\left(\frac{1}{6}-\xi\right)^2 R^2\\
&+\frac{1}{6}\left(\frac{1}{5}-\xi\right)\Box R - \left(\frac{1}{6}-\xi\right)RV - \frac{1}{6}\Box V + \frac{1}{2}V^2.
\end{aligned} \tag{4.41}$$

These coefficients can be derived recursively (see [26]) and the first coefficient a_0 is fixed by the boundary condition $h(x,y;0) = \delta^d(x-y)/\sqrt{|g|}$. One can immediately see that in the case of conformal coupling in four dimensions, $\xi = \frac{d-2}{4(d-1)} = \frac{1}{6}$, the first term in a_1 and two terms in a_2 drop out.
Now we want to turn to the discussion of regularising the effective action. We will show that one can get rid of the divergent terms by dimensional regularisation, where we will introduce corresponding counter terms to absorb the divergences. As we take the limit of $\tau_{UV}\to 0$, we

can expand the gamma functions in the divergent part of the effective action around $d = 4$ [15],

$$\begin{aligned}
\lim_{\tau_{UV}\to 0} \Gamma\left(-\frac{d}{2}, m^2\tau_{UV}\right) &= \frac{4}{d(d-2)}\left(\frac{2}{4-d} - \gamma\right) + \mathcal{O}(d-4),\\
\lim_{\tau_{UV}\to 0} \Gamma\left(1-\frac{d}{2}, m^2\tau_{UV}\right) &= \frac{2}{2-d}\left(\frac{2}{4-d} - \gamma\right) + \mathcal{O}(d-4),\\
\lim_{\tau_{UV}\to 0} \Gamma\left(2-\frac{d}{2}, m^2\tau_{UV}\right) &= \left(\frac{2}{4-d} - \gamma\right) + \mathcal{O}(d-4),
\end{aligned} \tag{4.42}$$

where $\gamma = 0.577$ is the Euler constant. We substitute this into the effective action to obtain

$$\Gamma_{div}[g_{\mu\nu}] = -\frac{1}{2(4\pi)^{d/2}}\left(\frac{m}{\mu}\right)^{d-4}\int d^dx\sqrt{|g|}\left(\frac{1}{d-4} + \frac{\gamma}{2}\right)\left[\frac{4m^4 a_0}{d(d-2)} + \frac{2m^2 a_1}{2-d} + a_2\right]. \tag{4.43}$$

This gives a closed form expression of the divergent terms in the effective action for $d = 4$.

4.3.3 Dimensional regularisation of the effective action

From the coefficients in eq. 4.41 one already sees that to absorb the divergences we must introduce higher derivative terms into the gravitational action. Hence, we write the extension of the Einstein-Hilbert action as [22],

$$S_g[g_{\mu\nu}] = -\frac{1}{16\pi G}\int d^dx\sqrt{|g|}\left(R + 2\Lambda + aR^2 + bR_{\alpha\beta}R^{\alpha\beta} + cR_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}\right), \tag{4.44}$$

where we have reintroduced Newtons gravitational constant G to highlight the regularisation. Our goal is to show that the sum $\Gamma^R_{div} + S_g$ is finite, where Γ^R_{div} includes the divergent curvature terms (including the a_0 term) of Γ_{div}. We can collect different curvature terms and combine them with the bare parameters $G_B, \Lambda_B, a_B, b_B, c_B$ to to absorb the divergences. Then we are left with the regularised and finite parameters G, Λ, a, b, c. Let us start by looking at the terms independent of curvature parameters. We absorb the divergence in the cosmological constant as

$$-\frac{1}{2(4\pi)^{d/2}}\left(\frac{m}{\mu}\right)^{d-4}\left(\frac{1}{d-4} + \frac{\gamma}{2}\right)\frac{4m^4}{d(d-2)} - \frac{2\Lambda_B}{16\pi G} = \frac{2\Lambda}{16\pi G}. \tag{4.45}$$

For the terms proportional to the Ricci scalar R, we absorb the divergence in the gravitational constant as

$$\frac{1}{2(4\pi)^{d/2}}\left(\frac{m}{\mu}\right)^{d-4}\left(\frac{1}{d-4} + \frac{\gamma}{2}\right)\frac{2m^2}{2-d}\left(\frac{1}{6} - \xi\right) - \frac{1}{16\pi G_B} = -\frac{1}{16\pi G}. \tag{4.46}$$

The remaining terms in our modified gravitational action are higher derivative terms of the metric, proportional to curvature parameters squared. The a_2 term in the heat kernel proper time expansion contains derivatives of the metric up to fourth order and we must regularize by absorbing corresponding divergences in the parameters a, b, c as

$$\begin{aligned}
-\frac{1}{2(4\pi)^{d/2}}\left(\frac{m}{\mu}\right)^{d-4}\left(\frac{1}{d-4} + \frac{\gamma}{2}\right)\frac{1}{2}\left(\frac{1}{6} - \xi\right)^2 - \frac{a_B}{16\pi G} &= -\frac{a_B}{16\pi G},\\
\frac{1}{180}\frac{1}{2(4\pi)^{d/2}}\left(\frac{m}{\mu}\right)^{d-4}\left(\frac{1}{d-4} + \frac{\gamma}{2}\right) - \frac{b_B}{16\pi G} &= -\frac{b}{16\pi G},\\
-\frac{1}{180}\frac{1}{2(4\pi)^{d/2}}\left(\frac{m}{\mu}\right)^{d-4}\left(\frac{1}{d-4} + \frac{\gamma}{2}\right) - \frac{c_B}{16\pi G} &= -\frac{c}{16\pi G}.
\end{aligned} \tag{4.47}$$

The term proportional to $\Box R$ in a_2 is a total derivative and therefore only gives a boundary contribution which we will ignore here.
The higher derivative terms, which we included for regularisation purposes will bring additions to the gravitational part of the field equations 4.16, namely [15, 22, 27],

$$\begin{aligned}
{}^{(1)}H_{\mu\nu} =& -\frac{1}{\sqrt{|g|}}\frac{\delta}{\delta g^{\mu\nu}}\int d^dx\sqrt{|g|}R^2 \\
=&\frac{1}{2}g_{\mu\nu}R^2 - 2RR_{\mu\nu} - 2g_{\mu\nu}\Box R + 2\nabla_\mu\nabla_\nu R, \\
{}^{(2)}H_{\mu\nu} =& -\frac{1}{\sqrt{|g|}}\frac{\delta}{\delta g^{\mu\nu}}\int d^dx\sqrt{|g|}R_{\alpha\beta}R^{\alpha\beta} \\
=&\frac{1}{2}g_{\mu\nu}R_{\alpha\beta}R^{\alpha\beta} - \Box R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\Box R + \nabla_\mu\nabla_\nu R - 2R^{\alpha\beta}R_{\alpha\mu\beta\nu} \\
=&2\nabla^\alpha\nabla_\nu R_{\mu\alpha} - \Box R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\Box R - 2R_{\mu\alpha}R^\alpha{}_\nu + \frac{1}{2}g_{\mu\nu}R_{\alpha\beta}R^{\alpha\beta}, \\
H_{\mu\nu} =& -\frac{1}{\sqrt{|g|}}\frac{\delta}{\delta g^{\mu\nu}}\int d^dx\sqrt{|g|}R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} \\
=&\frac{1}{2}g_{\mu\nu}R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} - 2R_{\mu\alpha\beta\gamma}R_\nu{}^{\alpha\beta\gamma} - 4\Box R_{\mu\nu} \\
&+ 2\nabla_\mu\nabla_\nu R + 4R_{\mu\alpha}R^\alpha{}_\nu - 4R^{\alpha\beta}R_{\alpha\mu\beta\nu}.
\end{aligned} \tag{4.48}$$

Not all of these terms are independent for $d = 4$. The generalised Gauss-Bonnet theorem [15, 22, 26] states that in four dimensions

$$\frac{\delta}{\delta g^{\mu\nu}}\int d^dx\sqrt{|g|}\left(R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} + R^2 - 4R_{\alpha\beta}R^{\alpha\beta}\right) = 0. \tag{4.49}$$

This imposes a relation between ${}^{(1)}H_{\mu\nu}, {}^{(2)}H_{\mu\nu}$ and $H_{\mu\nu}$ and lets us express

$$H_{\mu\nu} = -{}^{(1)}H_{\mu\nu} + 4{}^{(2)}H_{\mu\nu}. \tag{4.50}$$

In return, this imposes the condition $c = -a + 4b$ upon the coefficients of the higher derivative terms and we may freely choose $c = 0$.
Hence, we can write down an extended version of the semi-classical Einstein equation 4.16,

$$\mathbf{G}_{\mu\nu} - g_{\mu\nu}\Lambda = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + 8\pi G\left(a\,{}^{(1)}H_{\mu\nu} + b\,{}^{(2)}H_{\mu\nu}\right) - g_{\mu\nu}\Lambda = 8\pi G\langle T_{\mu\nu}\rangle, \tag{4.51}$$

where the parameters Λ, G, a, b are free parameters of our theory. These gravitational higher derivative terms may become dominant in regions of high curvature of our background. We cannot make predictions above the Planck scale. So we always consider length scales way above the Planck length, $l_p = \sqrt{G\hbar/c^3} \simeq 1.62 \times 10^{-33}$ cm, and therefore regions of relatively low curvature. As long as $|R| \gg \left|R^2\right|$ holds eq. 4.51 is a valid approximation and we can safely ignore even higher derivative terms. General relativity predicts experimental results to an incredible accuracy without introducing ${}^{(1)}H_{\mu\nu}$ and ${}^{(2)}H_{\mu\nu}$. So to keep consistency with experiment, the new terms in our gravitational field equations must be of very small magnitude compared to the original ones.
Next we should focus on the remaining terms, containing the potential V. Firstly, note that in the case of a conformally coupled field the term proportional to RV in a_2 (see eq.

4.41) will drop out. The last two terms proportional to $\Box V$ and V^2 must be regularised corresponding to the different types of the possible potential terms. For additional curvature terms a similar procedure to the one above must be used, where one introduces counter terms in the gravitational action, S_g, to compensate for the potential terms. Constant terms in V can be removed by redefining Λ/G, as in eq. 4.45.

4.4 The conformal anomaly

In this section we want to consider the case of a conformally invariant, i.e. scale invariant, theory and the effects of the quantum nature on the EMT expectation value. We will see that this leads to the well known *conformal* or *trace anomaly*, which has a wide range of effects in semi-classical gravity and many implications on cosmology.
We have seen that the FLRW metric is conformally flat,

$$ds^2 = a(\eta)(d\eta^2 - \delta_{ij}dx^i dx^j) = a(\eta)\eta_{\mu\nu}dx^\mu dx^\nu. \tag{4.52}$$

Especially, for the de Sitter metric in the Poincaré patch $a(\eta) = (H\eta)^{-2}$. For conformally invariant theories the classical action, $S[\chi, g_{\mu\nu}]$, is assumed to be invariant under conformal transformations

$$g_{\mu\nu}(x) \to \Omega^2(x) g_{\mu\nu}(x), \tag{4.53}$$

i.e. one assumes scale invariance of the theory at each point. Under conformal transformations the conformally coupled, massless equation of motion transforms as

$$\left(\Box - \frac{1}{4}\frac{d-2}{d-1}R\right)\phi \to \Omega^{-\frac{(d+2)}{2}}\left(\Box - \frac{1}{4}\frac{d-2}{d-1}R\right)\phi \tag{4.54}$$

and hence is invariant. The inclusion of a mass breaks conformal invariance, as one includes a parameter of fixed energy scale [15].
For conformally invariant theories the trace of the EMT is given by [15],

$$T[g_{\mu\nu}] = T^{\mu}{}_{\mu}[g_{\mu\nu}] = -\frac{\Omega}{\sqrt{|g|}}\frac{\delta S[\chi, \Omega^2 g_{\mu\nu}]}{\delta\Omega}\bigg|_{\Omega=1}, \tag{4.55}$$

from which it is immediately clear that $T = T^{\mu}{}_{\mu} = 0$ for a conformally invariant theory where $S[\chi, \Omega^2 g_{\mu\nu}] = S[\chi, g_{\mu\nu}]$.
Let us now try to calculate the trace of the quantum expectation value of the EMT, $T := \langle T^{\mu}{}_{\mu}\rangle$. We can make use of the effective action expansion, eq. 4.39,

$$\Gamma[g_{\mu\nu}] = \frac{1}{2(4\pi)^{d/2}}\left(\frac{m}{\mu}\right)^{d-4}\int d^dx\sqrt{|g|}\sum_{n=0}^{\infty} a_n(x) m^{4-2n}\Gamma\left(n - \frac{d}{2}; m^2\tau_{UV}, m^2\tau_{IR}\right), \tag{4.56}$$

to extract the UV divergent part of the effective action. Here $\Gamma(s; a, b) = \int_a^b dt\, t^{s-1}e^{-t}$. Above, in eq. 4.33, we used the non-zero mass as a regulator for the IR divergence of the effective action. This argument is no longer valid in the limit $m \to 0$ which is why we introduced an IR cutoff, τ_{IR}.
Nonetheless, we can use the above expansion to learn about the UV divergence of the theory.

In the $d = 4$ case, the terms $n = 0, 1, 2$ diverge. The $n = 0, 1$ terms vanish in the limit $m \to 0$ and therefore the only UV divergent term in the expansion is

$$\Gamma_{div}[g_{\mu\nu}] = \frac{1}{2(4\pi)^{d/2}} \left(\frac{m}{\mu}\right)^{d-4} \Gamma\left(2 - \frac{d}{2}\right) \int d^d x \sqrt{|g|}\, a_2(x). \tag{4.57}$$

For a conformally coupled, $\xi = \frac{1}{4}\frac{d-2}{d-1}$, massless scalar field the a_2 coefficient in eq. 4.41, becomes

$$a_2 = \frac{1}{180}\left(R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} - R_{\alpha\beta}R^{\alpha\beta}\right). \tag{4.58}$$

Here, we neglected two terms, the R^2 term vanishes in the limit $d \to 4$ due to the conformal coupling (also respecting the singularity of the Γ-function) and the $\Box R$ term is a total derivative and will therefore not contribute to the action [15]. One can write a_2 in terms of the the square of the Weyl tensor in $d = 4$,

$$U = C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} - 2R_{\alpha\beta}R^{\alpha\beta} + \tfrac{1}{3}R^2 \tag{4.59}$$

and the topological invariant quantity in eq. 4.49 [15],

$$W = R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} + R^2 - 4R_{\alpha\beta}R^{\alpha\beta}. \tag{4.60}$$

Then one finds

$$a_2 = \frac{1}{180}\left(\frac{3}{2}U - \frac{1}{2}W\right). \tag{4.61}$$

The point of doing this is that U and W (in $d = 4$) remain invariant under conformal transformations and hence Γ_{div} is invariant [15].
The UV divergent part of the EMT trace is given by

$$T_{div} = \frac{2}{\sqrt{|g(x)|}} g^{\mu\nu} \frac{\delta\, \Gamma_{div}[g_{\mu\nu}]}{\delta g^{\mu\nu}}. \tag{4.62}$$

We can first calculate [15]

$$\begin{aligned} \frac{2}{\sqrt{|g(x)|}} g^{\mu\nu} \frac{\delta}{\delta g^{\mu\nu}} \int d^d x \sqrt{|g|}\, a_2(x) &= \frac{1}{90}\left({}^{(1)}H^{\mu}{}_{\mu} - 3\,{}^{(2)}H^{\mu}{}_{\mu}\right) \\ &= -\frac{(d-4)}{180}\left(\frac{3}{2}U + \Box R - \frac{1}{2}W\right). \end{aligned} \tag{4.63}$$

Hence, the UV divergent part of the EMT trace is given by

$$T_{div} = -\frac{1}{2(4\pi)^{d/2}} \left(\frac{m}{\mu}\right)^{d-4} \Gamma\left(2 - \frac{d}{2}\right) \frac{(d-4)}{180} \left(\frac{3}{2}U + \Box R - \frac{1}{2}W\right). \tag{4.64}$$

We see that when taking $d \to 4$ the divergence in the Γ-function is cancelled by the $(d-4)$ factor. Making use of the expansion in 4.42, we find that

$$\begin{aligned} T_{div} &= \frac{1}{16\pi^2}\frac{1}{180}\left(\frac{3}{2}U + \Box R - \frac{1}{2}W\right) \\ &= \frac{1}{16\pi^2}\frac{1}{180}\left(R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} - R_{\alpha\beta}R^{\alpha\beta} + \Box R\right) \end{aligned} \tag{4.65}$$

which is independent of m and hence corresponds to the massless limit.
As $\Gamma[g_{\mu\nu}]$ is conformally invariant in the massless, conformally coupled case, one expects that for the total EMT, $T = 0$ [15]. Furthermore, given the above arguments, we find that the regularised trace component,

$$T_{reg} = T - T_{UV} = -T_{UV}, \tag{4.66}$$

is non-zero. In pure dS^4, the result reduces to

$$T_{reg} = \frac{1}{16\pi^2}\frac{1}{180}\frac{R^2}{12} = \frac{1}{16\pi^2}\frac{1}{15}H^4 \tag{4.67}$$

where we have used the identities in eq. 2.20. Other regularisation methods equivalently produce the trace anomaly [15]. For odd dimensions, the effective action in 4.39 is finite, and hence there is no trace anomaly [15].
This is a generally unexpected result. From the classical EMT one expects a similar behaviour of the quantum expectation value of the scalar field EMT. We expect this especially because both the full and the UV divergent effective actions remain conformally invariant in $d = 4$. This feature is known as the *trace anomaly* and stems from the non-conformal nature of T_{UV} away from $d = 4$, due to the divergent part of the effective action. This result is a good example how the quantum nature can have effects also on the classical level and lead to unexpected results.

4.5 The Bunch-Davies energy-momentum tensor

Finally, we are in the position to calculate the anticipated quantum expectation value of the EMT. We will start by calculating the *in-in* expectation value of the EMT and hence will be using the euclidean two point function in this section. Once again, let us consider a scalar field with mass m coupled to gravity with the action

$$S[\phi, g_{\mu\nu}] = \frac{1}{2}\int d^dx\sqrt{|g|}\left[g^{\mu\nu}\nabla_\mu\phi(x)\nabla_\nu\phi(x) - m^2\phi^2(x) + \xi R\phi^2(x)\right]. \tag{4.68}$$

From our definition of the expectation value of the EMT in eq. 4.15 we find that the quantity we wish to calculate is [15]

$$\begin{aligned}\langle T_{\mu\nu}\rangle =&(1-2\xi)\langle\nabla_\mu\phi\nabla_\nu\phi\rangle + \left(2\xi - \frac{1}{2}\right)g_{\mu\nu}g^{\alpha\beta}\langle\nabla_\alpha\phi\nabla_\beta\phi\rangle - 2\xi\langle\phi\nabla_\mu\nabla_\nu\phi\rangle + \frac{2}{d}\xi g_{\mu\nu}\langle\phi\Box\phi\rangle\\ &+\xi\left(R_{\mu\nu} - \tfrac{1}{2}Rg_{\mu\nu} + \frac{2(d-1)}{d}\xi R g_{\mu\nu}\right)\langle\phi^2\rangle + 2\left(\frac{1}{4} - \left(1 - \frac{1}{d}\right)\xi\right)m^2 g_{\mu\nu}\langle\phi^2\rangle,\end{aligned} \tag{4.69}$$

where the equation of motion $(\Box + m^2 - \xi R)\phi = 0$ was used for simplification. The two point function in the coincidence limit is invariant under the full de Sitter group, as the $i\epsilon$-prescription vanishes. This can alternatively be seen from the fact that the two point function equals the anticommutator and the Feynman Green function

$$\langle\phi^2(x)\rangle \equiv G_+(x,x) = G^{(1)}(x,x) = G_F(x,x) \tag{4.70}$$

in the coincidence limit and from sec. 3.7 we know that the latter two are invariant under $O(1, d)$.
We can make use of one very interesting property of maximally symmetric spaces. If we require that our resulting EMT is symmetric and invariant under the full de Sitter group, the only tensor satisfying these properties is the metric tensor (see F) [10]. Hence, the EMT of our scalar field must be proportional to the metric tensor

$$\langle T_{\mu\nu} \rangle = \frac{T}{d} g_{\mu\nu}, \tag{4.71}$$

where T is the trace of the EMT. Taking the trace of eq. 4.69, we find

$$T = 2(d-1)\left(\xi - \tfrac{1}{4}\tfrac{d-2}{d-1}\right) \Box G_+(x,x) + m^2 G_+(x,x), \tag{4.72}$$

where we have used the equation of motion and $g^{\alpha\beta}\langle \nabla_\alpha \phi \nabla_\beta \phi \rangle = \frac{1}{2}\Box\langle\phi^2\rangle - \langle \phi \Box \phi \rangle$. The EMT is directly proportional to the two point function $G_+(x,x)$ in the coincidence limit and derivatives thereof. Therefore, the only thing left to do is to find the regularised form of $G_+(x,x)$. To this end, let us take the coincidence limit of the euclidean two point function in eq. 3.63 [15, 28, 39],

$$G_+(x,x) = \frac{H^{d-2}}{(4\pi)^{d/2}} \frac{\Gamma\left(\frac{d-1}{2}+n\right)\Gamma\left(\frac{d-1}{2}-n\right)}{\Gamma\left(\frac{1}{2}+n\right)\Gamma\left(\frac{1}{2}-n\right)} \Gamma\left(1-\tfrac{d}{2}\right). \tag{4.73}$$

We see that this expression is independent of x and therefore $\Box G_+(x,x) = 0$ this simplifies our expression for the EMT to

$$T = m^2 \langle \phi^2 \rangle. \tag{4.74}$$

This expression is divergent for $d = 4$. From our calculation of the proper time expansion of the heat kernel we can extract the divergent parts of our two point function in the coincidence limit using eqs. 4.25, 4.30 and 4.35. To remove the divergences we must expand eq. 4.73 around $d = 4$ and then subtract terms from the proper time expansion up to the correct order [15]. From eqs. 4.25 and 4.30 we find that

$$\zeta(1) = \int d^d x \sqrt{|g|}\, i G_F(x,x) = \int d^d x \sqrt{|g|}\, i G_+(x,x). \tag{4.75}$$

Then from eq. 4.35 we find that

$$G_+(x,x) = \frac{1}{(4\pi)^{d/2}} \sum_n a_n(x) m^{d-2(n+1)} \Gamma\left(n+1-\tfrac{d}{2}, m^2\tau_{UV}\right). \tag{4.76}$$

The terms to be subtracted from our divergent result should be up to the same order in the proper time expansion as we used to renormalise the effective action. Otherwise we will again obtain a divergence when trying to calculate the effective action with the obtained result [15]. Hence, in analogy to the previous section we can extract the terms up to second order

$$\begin{aligned} G_+^{div}(x,x) = \frac{m^{d-2}}{(4\pi)^{d/2}} \Bigg[& a_0(x)\Gamma\left(1-\tfrac{d}{2}, m^2\tau_{UV}\right) \\ & + \frac{a_1(x)}{m^2}\Gamma\left(2-\tfrac{d}{2}, m^2\tau_{UV}\right) + \frac{a_2(x)}{m^4}\Gamma\left(3-\tfrac{d}{2}, m^2\tau_{UV}\right) \Bigg]. \end{aligned} \tag{4.77}$$

The expansion parameters a_0, a_1, a_2 are given in eqs. 4.41 and simplify greatly in de Sitter space. Setting $V = 0$ in eqs. 4.41 and using the results from sec. 2,

$$\begin{aligned} a_0 =& 1, \\ a_1 =& -\left(\frac{1}{6}-\xi\right) R = -d(d-1)\left(\frac{1}{6}+\xi\right) H^2, \\ a_2 =& \frac{1}{180}\left(R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} - R_{\alpha\beta}R^{\alpha\beta}\right) + \frac{1}{2}\left(\frac{1}{6}-\xi\right)^2 R^2 + \tfrac{1}{6}\left(\frac{1}{5}-\xi\right)\Box R \\ =& \left[\frac{1}{2}\left(\frac{1}{6}-\xi\right)^2 - \frac{1}{180}\frac{d-3}{d(d-1)}\right] d^2(d-1)^2 H^4. \end{aligned} \tag{4.78}$$

Expanding eq. 4.73 around $d = 4$, we find

$$\begin{aligned} G_+(x,x) = \frac{H^2}{16\pi^2}\Bigg[&\left(\frac{m^2}{H^2} - 12\left(\frac{1}{6}-\xi\right)\right) \\ &\times\left(\frac{2}{d-4} + \psi\left(\tfrac{3}{2}-n\right) + \psi\left(n+\tfrac{3}{2}\right) + \ln\left(\frac{H^2}{4\pi}\right) + \gamma - 1\right) \\ &+ 24\xi' + 14\xi - 3\Bigg] + \mathcal{O}(d-4) \end{aligned} \tag{4.79}$$

where $\gamma = 0.577$ is the Euler constant and $\psi(z) = \Gamma'(z)/\Gamma(z)$ is the digamma function, $R = -12H^2$ and $\xi' = \frac{\mathrm{d}\xi}{\mathrm{d}d}$. Similarly, expanding G_+^{div} and using the results in eqs. 4.42 we find

$$\begin{aligned} G_+^{div}(x,x) = \frac{H^2}{16\pi^2}\Bigg[&\left(\frac{m^2}{H^2} - 12\left(\frac{1}{6}-\xi\right)\right)\left(\frac{2}{d-4} + \ln\left(\frac{m^2}{4\pi}\right) + \gamma - 1\right) \\ &- 26\left(\frac{1}{6}-\xi\right) + 24\xi'\Bigg]. \end{aligned} \tag{4.80}$$

Now we can subtract the divergent parts of the two point function from the full expression, leading to the regularised and finite result

$$\begin{aligned} G_+(x,x) - G_+^{div}(x,x) = \frac{H^2}{16\pi^2}\Bigg[&\left(\frac{m^2}{H^2} - 12\left(\frac{1}{6}-\xi\right)\right) \\ &\times\left(\psi\left(\tfrac{3}{2}-n\right) + \psi\left(\tfrac{3}{2}+n\right) - \ln\left(\frac{m^2}{H^2}\right) - 1\right) \\ &+ \frac{m^2}{H^2} - \frac{2}{3} - \frac{a_2}{m^2H^2}\Bigg]. \end{aligned} \tag{4.81}$$

In combination with the above results in eqs. 4.71, 4.74 and 4.78 we find a closed form expression for the regularised *in-in* EMT expectation value of a massive scalar field in dS^4,

$$\begin{aligned} \langle T_{\mu\nu}\rangle = -g_{\mu\nu}\frac{m^2H^2}{64\pi^2}\Bigg[&\left(12\left(\frac{1}{6}-\xi\right) - \frac{m^2}{H^2}\right)\left(\psi\left(\tfrac{3}{2}-n\right) + \psi\left(\tfrac{3}{2}+n\right) - \ln\left(\frac{m^2}{H^2}\right)\right) \\ &+ \frac{2}{3} - 12\left(\frac{1}{6}-\xi\right) + 72\left(\left(\frac{1}{6}-\xi\right)^2 - \frac{1}{1080}\right)\frac{H^2}{m^2}\Bigg], \end{aligned} \tag{4.82}$$

which is in agreement with [15] and [28]. From this we can also extract values for energy density

$$\rho = \langle T^{00} \rangle = \frac{T}{d}. \tag{4.83}$$

In the conformally coupled case we have $\xi = \frac{1}{6}$ and the result simplifies to

$$\langle T_{\mu\nu} \rangle = -g_{\mu\nu} \frac{m^2 H^2}{64\pi^2} \left[-\frac{m^2}{H^2} \left(\psi\left(\tfrac{3}{2} - n\right) + \psi\left(\tfrac{3}{2} + n\right) - \ln\left(\frac{m^2}{H^2}\right)\right) + \frac{2}{3} - \frac{1}{15}\frac{H^2}{m^2} \right]. \tag{4.84}$$

Furthermore, in the massless limit we have

$$\langle T_{\mu\nu} \rangle = g_{\mu\nu} \frac{9H^4}{8\pi^2} \left[\frac{1}{2}\left(\frac{1}{6} - \xi\right) + \frac{1}{1080} - \left(\frac{1}{6} - \xi\right)^2 \right]. \tag{4.85}$$

Lastly, in the massless and conformally coupled case

$$\langle T_{\mu\nu} \rangle = g_{\mu\nu} \frac{1}{64\pi^2} \frac{H^4}{15} = -g_{\mu\nu} \frac{a_2}{64\pi^2}, \tag{4.86}$$

reproducing the trace anomaly as expected [15].

4.6 The contribution to the Einstein equation

Now we want to turn to analysing our result of eq. 4.82 in the light of the semi-classical Einstein equation. We have already discussed that adding a matter contribution to pure de Sitter space will result in inconsistencies as we do not have any freedom in the metric to take the backreaction of the massive field into account. Nonetheless, we have seen that the EMT contribution is proportional to the metric and can therefore be viewed as a simple screening of the cosmological constant.
Let us consider again the semi-classical Einstein equation from eq. 4.16 or 4.51,

$$\mathbf{G}_{\mu\nu} = g_{\mu\nu}\Lambda + 8\pi \langle T_{\mu\nu} \rangle. \tag{4.87}$$

We see that indeed the backreaction on the spacetime is such that it can oppose or enhance the cosmological constant depending on the overall sign of the EMT.
Before we analyse eq. 4.82 let us consider the results for a classical scalar field for comparison. The EMT of a classical scalar field ϕ_{cl} is [1]

$$T^{\mu\nu}_{cl} = \partial^\mu \phi_{cl} \partial^\nu \phi_{cl} - g^{\mu\nu} \left(\partial^\alpha \phi_{cl} \partial_\alpha \phi_{cl} + V(\phi_{cl}) \right), \tag{4.88}$$

where $V(\phi_{cl})$ is an arbitrary potential. Let us for simplicity assume that the field only depends on time, i.e. $\phi_{cl}(x) = \phi_{cl}(\eta)$. Then the contribution to the energy density is

$$T^{00}_{cl} = \frac{1}{2} \partial^0 \phi_{cl} \partial^0 \phi_{cl} - g^{00} V(\phi_{cl}). \tag{4.89}$$

Therefore, as the potential increases in the positive direction, we get a growing negative contribution to the energy density and hence oppose the action of the cosmological constant.

In the above section we have used the action in eq. 4.68 to obtain the EMT. Considering the same action, but for a classical field, the potential is (see eq. 3.15)

$$V(\phi_{cl}) = \frac{1}{\eta^2}\left(\frac{m^2}{H^2} + 12\left(\xi - \frac{1}{6}\right)\right)\phi_{cl}^2. \tag{4.90}$$

So only for small values of m^2 and values of ξ below the conformal coupling, where the potential energy is small, we should get a positive contribution to the energy density.
To check our expectations, let us take large and small mass limits of eq. 4.82. In the large mass limit $\frac{m^2}{H^2} \gg 1$ we have

$$\begin{aligned}\psi\left(\tfrac{3}{2} - n\right) + \psi\left(\tfrac{3}{2} + n\right) - \ln\left(\frac{m^2}{H^2}\right) &= \ln\left(\frac{H^2}{m^2}\right) + \ln\left(\frac{m^2}{H^2}\right) - \frac{4}{3}\frac{H^2}{m^2} + \mathcal{O}\left(\frac{H^3}{m^3}\right) \\ &= -\frac{4}{3}\frac{H^2}{m^2} + \mathcal{O}\left(\frac{H^3}{m^3}\right)\end{aligned} \tag{4.91}$$

so that we find

$$\langle T^{00}\rangle \approx -g^{00}\frac{m^2 H^2}{64\pi^2}\xi < 0, \tag{4.92}$$

which is strictly negative. This definitely confirms our expectations. In this case the potential is large and strictly positive and hence the contribution to the EMT is strictly negative.
In the limit of small mass $\frac{m^2}{H^2} \ll 1$, we can approximate the EMT as

$$\langle T^{\mu\nu}\rangle = g^{\mu\nu}\frac{9H^4}{8\pi^2}\left[\frac{1}{2}\left(\frac{1}{6} - \xi\right) + \frac{1}{1080} - \left(\frac{1}{6} - \xi\right)^2\right]. \tag{4.93}$$

Therefore, depending on the coupling parameter, we have

$$\begin{aligned}\langle T^{00}\rangle &> 0 \quad \text{for} \quad 0 < \xi < \frac{1}{180}\left(\sqrt{2055} - 15\right) \approx 0.1685, \\ \langle T^{00}\rangle &< 0 \quad \text{for} \quad \frac{1}{180}\left(\sqrt{2055} - 15\right) < \xi.\end{aligned} \tag{4.94}$$

So the sign of the contribution flips just below the value for conformal coupling, $\xi \approx 0.1666$. For a value below the conformal coupling, the potential can actually become negative if the mass is sufficiently small. Then even the potential gives a positive contribution to the energy density. On the other hand, once we take the coupling to be larger than the conformal value the potential grows and we also get a growing negative contribution to the energy density. So this is also in agreement with our expectation.

4.7 The energy-momentum tensor for general α-vacua

Up to now we have only calculated the BD contribution to the EMT. The obtained result in eq. 4.82 is not the most general as we have made use of the boundary conditions leading to the *in-in* expectation value. In sec. 3.7 we found that out of the class of (α, β)-vacua we obtain $O(1, d)$ invariance of the anticommutator and Feynman two point functions only if we

set $\beta = 0$ and restrict to the one parameter class of α-vacua [1]. We also used the assumption of full de Sitter invariance to simplify the EMT to its trace in eq. 4.71. Therefore let us also restrict to only the class of α-vacua in this section and construct the most general EMT in this case.
Consider the result, eq. 3.74, in the coincidence limit [2],

$$\begin{aligned} G_+^{(\alpha,0),(\alpha',0)}(1) = \frac{1}{\gamma^*}\Big[&\left(\cosh\alpha\cosh\alpha' + \sinh\alpha\sinh\alpha'\right) G_+(1) \\ &+ \left(\cosh\alpha\sinh\alpha' + \sinh\alpha\cosh\alpha'\right) G_+^E(-1)\Big], \\ \gamma^* = &\cosh\alpha\cosh\alpha' - \sinh\alpha\sinh\alpha'. \end{aligned} \tag{4.96}$$

As discussed below eq. 3.74, the term proportional to $G_+(-1)$ is just a constant and we can write

$$\begin{aligned} G_+^{(\alpha,0),(\alpha',0)}(1) &= AG_+^E(1) + B, \\ A &= \frac{\cosh\alpha\cosh\alpha' + \sinh\alpha\sinh\alpha'}{\cosh\alpha\cosh\alpha' - \sinh\alpha\sinh\alpha'}, \\ B &= A\ \frac{H^2}{(4\pi)^2}\Gamma\left(N_-\right)\Gamma\left(N_-\right) \end{aligned} \tag{4.97}$$

Taking into account eq. 4.74, we can immediately see that the general dS^4 EMT is

$$\langle T_{\mu\nu}\rangle^{(\alpha,0),(\alpha',0)} = A\left[\langle T_{\mu\nu}\rangle + \frac{m^2}{4}\frac{H^2}{(4\pi)^2}\Gamma\left(N_-\right)\Gamma\left(N_-\right) g_{\mu\nu}\right], \tag{4.98}$$

where $\langle T_{\mu\nu}\rangle$ is given by eq. 4.82. Hence upon Bogoliubov transformation the EMT only scaled by a certain value and shifted by a constant depending on α, α' and a constant coefficient. Note that B is strictly positive and therefore the additional term always enhances the cosmological constant, when it is non-zero.
Let us now consider specific choices of vacua. With the parameters α and α' we can control the choice of vacuum that we impose on our result. Especially common are the vacuum choices which diagonalise the Hamiltonian either in the asymptotic future or past, as discussed in sec. 3.4. Then possible choices correspond to *in-in-*, *in-out-* and *out-out-*vacua:

- The *in-in* expectation value corresponds to the choice $\alpha = \alpha' = 0$. In this case our EMT,

$$\langle T_{\mu\nu}\rangle_{in-in} = \langle T_{\mu\nu}\rangle \tag{4.99}$$

  corresponds to the euclidean result.

[1] On the other hand, in the EMT only two point functions in the coincidence limit enter, which are completely independent of any coordinates and therefore invariant under the full de Sitter group in any way.

[2] Note that in the coincidence limit $y \to x$, or $Z \to 1$, we see from eq. 3.63 that the euclidean Green function reduces to constant terms of the form

$$\lim_{y\to x} G_+^E(x,y) = G_+^E(-1) \sim {}_2F_1(a,b;c,0) = 1. \tag{4.95}$$

In this case we can also set $i\epsilon$ to zero, because we do not run into any singularity of any sort.

- The *in-out* expectation value corresponds to the choice $\alpha = 0$ and $\cosh\alpha' = \sinh\alpha'$, giving $\gamma^* = \cosh\alpha'$ and hence $A = B = 1$. The EMT corresponding to this choice is

$$\langle T_{\mu\nu}\rangle_{in-out} = \langle T_{\mu\nu}\rangle + \frac{m^2}{4} g_{\mu\nu}. \tag{4.100}$$

- The *out-out* expectation value corresponds to the choice $\cosh\alpha = \sinh\alpha$ and $\cosh\alpha' = \sinh\alpha'$, giving $\gamma^* = 0$ and hence the coefficients A, B diverge in this case

$$\langle T_{\mu\nu}\rangle_{out-out} \to \infty. \tag{4.101}$$

  This is a direct consequence of the fact that in the class of (α, β)-vacua the Bogoliubov coefficients are mode-independent and we generate an infinite energy density with respect to the BD vacuum in the asymptotic future, as discussed in sec. 3.4. Therefore, it is sensible that EMT in this case diverges and would cause singular backreaction.

In conclusion, as long as we respect de Sitter isometry, we will only get an EMT contribution which equals a constant times the metric (see F). This contribution can be seen as a shift of the cosmological constant. For a measure of the backreaction, de Sitter isometry must be broken explicitly
Nonetheless, our discussion has beard some very valuable fruits for cosmology. It turns out, that the higher derivative terms in the gravitational action give rise a non-eternal de Sitter solution, which supplies us with a natural mechanism of a graceful exit from the exponential expansion of the universe.

Chapter 5

Semi-classical gravity and inflation

In this section we want to have a further look into the meaning of the higher curvature terms, we added to the gravitational action due to regularisation purposes. On a historic note, these terms have turned out to be of great importance from a cosmological viewpoint. Starobinsky noticed their importance and gave birth to what has become known as *Starobinsky inflation*. Originally, this topic was discussed in what has become one of the most famous papers in the field of cosmology [29]. Here we want to discuss some of his findings in sec. 5.1 [10, 15, 22, 29]. Furthermore, in sec. 5.2 [1, 30] we turn to a more modern view of inflation and draw some parallels to Starobinsky inflation.

5.1 Starobinsky inflation

Recall the semi-classical equation from eq 4.51. Let us consider a FLRW universe with the metric in 2.1, in terms of conformal time[1] $dt = a(\eta)d\eta$. Since we are specifying to a conformally flat spacetime, we may investigate the second order curvature terms more closely. One finds that there is a simple relation between the newly added terms [15],

$$^{(1)}H_{\mu\nu} = 3\,^{(2)}H_{\mu\nu}, \tag{5.1}$$

so that we can choose to set $b = 0$ without loss of generality.
We want to consider the case of most general modified Einstein gravity up to second order in curvature expressions. So let us investigate the possibility of additional conserved tensors. In conformally flat space the Riemann tensor is given completely in terms of the Ricci and the metric tensors [10]

$$\begin{aligned} R_{\alpha\beta\gamma\delta} =& \frac{1}{d-2}\left(g_{\alpha\gamma}R_{\beta\delta} + g_{\beta\delta}R_{\alpha\gamma} - g_{\alpha\delta}R_{\beta\gamma} - g_{\beta\gamma}R_{\alpha\delta}\right) \\ &+ \frac{1}{(d-2)(d-1)}\left(g_{\alpha\delta}g_{\beta\gamma} - g_{\alpha\gamma}g_{\beta\delta}\right) \end{aligned} \tag{5.2}$$

as the Weyl tensor $C_{\alpha\beta\gamma\delta}$ vanishes here. Hence, with some additional considerations (see [15]) one finds six independent quantities with the correct units of $(length)^{-4}$,

$$R_{\mu\alpha}R^{\alpha}{}_{\nu},\ RR_{\mu\nu},\ \nabla_\mu\nabla_\nu R,\ g_{\mu\nu}R_{\alpha\beta}R^{\alpha\beta},\ g_{\mu\nu}R^2,\ g_{\mu\nu}\Box R, \tag{5.3}$$

[1] In this section we will consider the standard range of $-\infty < \eta < 0$.

out of which one can build covariantly conserved tensors.
We find that in addition to ${}^{(1)}H_{\mu\nu} = 3$ and ${}^{(2)}H_{\mu\nu}$, there is one more term that can be added to 4.51 which is only conserved in conformally flat spacetimes [15, 22],

$$\begin{aligned} {}^{(3)}H_{\mu\nu} =& - R_{\mu\alpha}R^{\alpha}{}_{\nu} + \tfrac{2}{3}RR_{\mu\nu} + \frac{1}{2}g_{\mu\nu}R_{\alpha\beta}R^{\alpha\beta} - \tfrac{1}{4}R^2 g_{\mu\nu} \\ =& - \frac{1}{12}R^2 g_{\mu\nu} + R^{\alpha\beta}R_{\alpha\mu\beta\nu}. \end{aligned} \tag{5.4}$$

This tensor is cannot be derived from variation of a curvature term in the action and is therefore only accidentally conserved in conformally flat spacetimes [15]. Including this new term, the semi-classical Einstein equation becomes

$$\mathfrak{G}_{\mu\nu} - g_{\mu\nu}\Lambda = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + 8\pi\left(\alpha^{(1)}H_{\mu\nu} + \beta^{(3)}H_{\mu\nu}\right) - g_{\mu\nu}\Lambda = 8\pi\langle T_{\mu\nu}\rangle. \tag{5.5}$$

Following the famous paper of Starobinsky [29], let us consider this equation for an empty universe with $\langle T_{\mu\nu}\rangle = 0$. The equations of motion for $a(\eta)$ are given by $\mathfrak{G}^{\mu}{}_{\nu} - \delta^{\mu}{}_{\nu}\Lambda = 0$. The individual components are [2]

$$\begin{aligned} \mathfrak{G}^0{}_0 =& \frac{3}{a^2}\left(\frac{a'^2}{a^2} + K\right) - \Lambda + \frac{24\pi\beta}{a^4}\left(\frac{a'^2}{a^2} + K\right)^2 \\ & - \frac{144\pi\alpha}{a^4}\left(4\frac{a'^2}{a^2}\frac{a''}{a} + \frac{a''^2}{a^2} + 2K\frac{a'^2}{a^2} - 2\frac{a'''}{a}\frac{a'}{a} - K^2\right), \\ \mathfrak{G}^1{}_1 =& \mathfrak{G}^2{}_2 = \mathfrak{G}^3{}_3, \\ \mathfrak{G}^{\mu}{}_{\mu} =& \frac{6}{a^3}(a'' + Ka) - 4\Lambda + \frac{96\pi\beta}{a^8}\left[aa'^2a'' - a'^4 + Ka^3a'' - Ka^2a'^2\right] \\ & + \frac{96\pi\alpha}{a^8}\left[3a^3a'''' - \frac{1}{8\pi}a'''a' + 18aa'^2a'' - 9a^2a''^2 - 6K\left(a^3a'' - a^2a'^2\right)\right] \end{aligned} \tag{5.6}$$

where $a' := \frac{\mathrm{d}\,a}{\mathrm{d}\eta}$. Note that the spatial and trace equations are of fourth order in derivatives of the scale factor, whereas the $0-0$ equation is only of third order. The general solutions to the 0-0 equation (in terms of t) are discussed in [29]. For simplicity let us only discuss the result for $K = 0$ here. The above equation admits the de Sitter solution

$$a(\eta) = -\frac{1}{H\eta}, \tag{5.7}$$

where the coefficient H obeys

$$-\frac{\Lambda}{3} + H^2 + 8\pi\beta H^4 = 0. \tag{5.8}$$

For a physical solution this sets the requirement that $\beta < 0$ as otherwise H becomes imaginary. We also see that we can safely set $\Lambda = 0$ and we still obtain a de Sitter solution with $H = 1/\sqrt{8\pi\,|\beta|}$. For $K = \pm 1$ one also obtains the respective de Sitter solution for the scale factor [29].
Therefore, from the second order curvature terms in the Einstein equation, one naturally

[2] The $0-0$ equation is equivalent to eq. (4) in [29], when substituting $\frac{\mathrm{d}}{\mathrm{d}t} \to \frac{1}{a(\eta)}\frac{\mathrm{d}}{\mathrm{d}\eta}$ and letting $k_2 = -2880\pi^2\beta$ and $k_3 = -17280\pi^2\alpha$.

obtains an expanding universe, when homogeneity and isotropy of the spacetime are imposed (FLRW). This is an amazing result since it removes some the arbitrariness of the de Sitter spacetime choice for a cosmological setting. But we can push it even further. We know that from the viewpoint of cosmology that we need to exit the exponentially expanding phase of the universe at some point. We require what is known as a *graceful exit* to a FLRW universe, where radiation or matter energy density can also play their dominating role [1].
With this in mind, let us now consider what happens if we allow for perturbations. We can consider the solution for the scale factor to be

$$a(\eta) = -\frac{1}{H\eta}\left(1 + sf(\eta)\right), \qquad s \ll 1. \tag{5.9}$$

where $f(\eta)$ is some perturbation within the class of FLRW universes. We can then substitute this into the 0-0 component of eq. 5.6 and solve perturbatively in s. To $\mathcal{O}\left(s^0\right)$ we just obtain the de Sitter solution from above. To $\mathcal{O}\left(s\right)$ we find the differential equation

$$6\alpha\eta^2\left(\eta f''' + f''\right) - (24\alpha - \beta)\eta f' + \beta f = 0. \tag{5.10}$$

The general solution to this equation is

$$f(\eta) = \frac{C_1}{\eta} + C_2\eta^{\frac{3}{2}-\sqrt{\frac{9}{4}-\frac{\beta}{6\alpha}}} + C_3\eta^{\frac{3}{2}+\sqrt{\frac{9}{4}-\frac{\beta}{6\alpha}}}, \tag{5.11}$$

In the inflationary paradigm the universe is in a de Sitter stage for the first moments and then transitions into a FLRW universe. This gives us reason to match this solution to the $\mathcal{O}\left(s^0\right)$ solution at the infinite past ($\eta \to -\infty$). It most sensible from a cosmological standpoint to first have an inflationary model with an early de Sitter stage, which then transfers to a FLRW type universe.
The parts of the perturbation which vanish at past infinity are

$$f(\eta) = \frac{C_1}{\eta} + C_2\eta^{\frac{3}{2}-\sqrt{\frac{9}{4}-\frac{\beta}{6\alpha}}}, \tag{5.12}$$

since we said that $\beta < 0$. But note that as we progress into the future these solutions will grow and become dominant. We therefore exit the perturbative regime. By analysing $\mathfrak{G}^0{}_0 = 0$ further, Starobinsky found that the de Sitter solution is an unstable saddle point and that non-singular deviations exist. Therefore, in this model we can realise the transition from the de Sitter stage into a FLRW universe [29].

5.2 Slow roll inflation

Above we have seen that including higher derivative terms in the gravitational action naturally gives us the freedom to produce a non-eternal de Sitter stage. The reason for this is that by introducing these higher order curvature terms we also introduce new degrees of freedom. One of these is a scalar degree of freedom which drives inflation [1].
Let us develop these ideas further. We will show that Starobinsky inflation can also be realised by introducing a scalar field with certain restrictions on its potential. Consider a general scalar field with potential $V(\varphi)$,

$$S = \int d^dx\sqrt{|g|}\left(\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - V(\varphi)\right). \tag{5.13}$$

in a FLRW universe. This gives the general equation of motion

$$\Box\varphi + V_{,\varphi} = \ddot{\varphi} + (d-1)H\dot{\varphi} - \frac{1}{a^2}\Delta\varphi + V_{,\varphi} = 0, \tag{5.14}$$

where $\dot{(.)} \equiv \frac{\mathrm{d}}{\mathrm{d}t}$ and $H = \dot{a}/a$. We can drop the spatial derivative term $\Delta\varphi = 0$ for spatially homogeneous and isotropic initial conditions, justified by the cosmological principle. By comparing the EMT of a hydrodynamical fluid with the one of a scalar field one obtains that energy density and pressure are respectively given by

$$\rho = \frac{1}{2}\dot{\varphi}^2 + V \quad \text{and} \quad p = \frac{1}{2}\dot{\varphi}^2 - V. \tag{5.15}$$

We can realize the equation of state of de Sitter, $\rho \approx -p$, by enforcing the condition $\frac{1}{2}\left|\dot{\varphi}^2\right| \ll |V|$. In this case the first Friedmann equation (see eqs. 2.8) then becomes

$$H = \frac{\dot{a}}{a} \approx \sqrt{\frac{16\pi V}{(d-2)(d-1)}}. \tag{5.16}$$

For an approximate de Sitter solution $H \approx const.$, which sets the requirement that the slow-roll parameters [30]

$$\epsilon = -\frac{\dot{H}}{H^2}, \qquad \delta = -\frac{\ddot{H}}{2H\dot{H}}, \tag{5.17}$$

are both small. In the slow-roll regime we can neglect the acceleration term in the scalar field equation of motion, resulting in

$$(d-1)H\dot{\varphi} + V_{,\varphi} \approx 0. \tag{5.18}$$

Combining this with eq. 5.16 we obtain the scale factor solution

$$a(\varphi) \approx a_0 \exp\left(\frac{16\pi}{d-2}\int_{\varphi}^{\varphi_0} d\tilde{\varphi}\,\frac{V}{V_{,\tilde{\varphi}}}\right), \tag{5.19}$$

giving the desired exponential expansion.
The conditions we used to obtain this solution,

$$\frac{1}{2}\left|\dot{\varphi}^2\right| \ll |V|, \qquad |\ddot{\varphi}| \ll (d-1)\,|H\dot{\varphi}| \approx |V_{,\varphi}|, \tag{5.20}$$

are known as slow roll conditions. We can reformulate them only in terms of the potential V as [1]

$$\left(\frac{V_{,\varphi}}{V}\right)^2 \ll 1, \qquad \left|\frac{V_{,\varphi\varphi}}{V}\right| \ll 1. \tag{5.21}$$

This is equivalent to $\epsilon \ll 1$ and $\delta \ll 1$. Then as soon as we violate these conditions, the inflationary stage of the universe ends and we model a graceful exit.
In conclusion, we have successfully showed that by introducing a scalar degree of freedom we can produce results which are equivalent to Starobinsky inflation [1].

Chapter 6

Loop corrections and IR effects in interacting theories

So far we have focused on free fields and UV effects of non-interacting scalar field theories. What we have learned so far is that the BD vacuum is a good choice for a vacuum in the UV limit, but is certainly not the only valid choice. Accepting that the BD solutions provide a good *in*-vacuum choice, we can introduce a formalism known as the *Schwinger-Keldysh*, *in-in* or *closed time path* (CTP) formalism which was designed as an extension to the Feynman diagrammatic technique for dynamic backgrounds and non-equilibrium processes [40]. In this framework one can preform calculations only with respect to the *in*-vacuum, making no reference to time evolved vacua.
We want to then turn away from the UV limit and use this formalism to discuss self-interactions of our field and effects that occur on larger scales.
Firstly, we want to give an overview of the IR limit of our previous results in sec. 6.1 [4, 38]. Then we want to introduce the CTP formalism briefly in sec. 6.2 [1, 8, 9, 15, 23, 31–37], before turning to a direct application and calculating loop corrections to the propagator in the IR limit in sec. 6.3 [5, 8, 9, 31, 37]. The found results will turn out to be de Sitter invariant and in sec. 6.4 [8] we will discuss the generality of our findings.

6.1 A naive estimate of the IR contributions at tree level

Consider the IR expansion of the BD modes in eq. 3.39 in the IR limit,

$$\begin{gathered} q_k(\eta) = \eta^{\frac{d-1}{2}} \left[A_-(k\eta)^{-n} + A_+(k\eta)^n + B(k\eta)^{2-n} + \mathcal{O}\left((k\eta)^{2+n}\right)\right], \quad \text{where} \\ A_- = -\frac{i2^n\Gamma(n)}{\pi}, \quad A_+ = -\frac{i2^{-n}\cos(\pi n)\Gamma(-n)}{\pi} + \frac{2^{-n}}{\Gamma(n+1)}, \quad B = -\frac{i2^{n-2}\Gamma(n)}{\pi(n-1)}. \end{gathered} \tag{6.1}$$

Therefore the leading contributions to the BD two point functions in the IR limit are

$$\langle\phi_{\boldsymbol{k}}(\eta_x)\phi_{\boldsymbol{k}}(\eta_y)\rangle \approx (\eta_x\eta_y)^{\frac{d-1}{2}} \left[|A_-|^2 (k\eta_x k\eta_y)^{-n} + A_-^* A_+ \left(\frac{\eta_y}{\eta_x}\right)^n + A_- A_+^* \left(\frac{\eta_x}{\eta_y}\right)^n\right]. \tag{6.2}$$

The first term is by far the most contributing in the IR limit, so let us focus on this term only. Transforming this back to position space, we find

$$\langle\phi(x)\phi(y)\rangle \propto (\eta_x\eta_y)^{\frac{d-1}{2}-n} \int \frac{\text{đ}^{d-1}k}{k^{2n}} e^{i\boldsymbol{k}.(\boldsymbol{x}-\boldsymbol{y})}. \tag{6.3}$$

Recall that $n = \sqrt{\left(\frac{d-1}{2}\right)^2 - \frac{m^2}{H^2}}$ and hence the behaviour depends on the magnitude of the mass of our field with respect to the curvature scale. For the the complementary series $\frac{m}{H} \ll \frac{d-1}{2}$ and $2n \approx d-1$. In this case we obtain a logarithmic IR divergence in the two point function. We would have to introduce an IR cutoff and could consider the consequences this has on cosmology [38]. But in this quasi massless limit one runs into even deeper problems, since then we lose the notion of a de Sitter invariant vacuum [4]. Therefore, we will keep the mass at a non-zero but small value, such that n remains real. Then also $d-2 > n$ and the above integral will converge at the lower bound. Using the intermediate result of sec. D, we can estimate

$$\int_0^\pi d\theta\ \sin^{d-3}\theta e^{-ikrcos\theta} \approx (kr)^{-\frac{d-3}{2}} J_{\frac{d-3}{2}}(kr) \sim (kr)^{-\frac{d-3}{2}} (kr)^{\frac{d-3}{2}} \sim \mathcal{O}(1), \tag{6.4}$$

where $r = |\boldsymbol{x} - \boldsymbol{y}|$. So this does not give any contribution to our estimate. Then we find for the complementary series that

$$\langle\phi(x)\phi(y)\rangle \propto (\eta_x\eta_y)^{\frac{d-1}{2}-n} \int dk\, k^{d-2-2n}, \tag{6.5}$$

which behaves smoothly in the IR limit of the integral, but diverges at the asymptotic future $\eta \to 0$.

6.2 The closed time path formalism

In most curved spacetimes we are dealing with a dynamical background and hence a system out of thermal equilibrium. The closed time path (CPT) formalism[1] is a very useful tool for studying such situations. We have observed that the BD vacuum is a valid candidate for the *in*-vacuum, but on the same footing does not diagonalize the Hamiltonian on later time slices. This alone is a very convincing argument for using a method which only makes reference to the *in*-vacuum. Here, we want to give a summary and short introduction to the method on the basis of [8, 31–35].

In quantum mechanics, time evolution from some time t_i to a later time t_f is given by the evolution operator $U(t_f, t_i)$ which is defined as the solution to

$$i\frac{\mathrm{d}}{\mathrm{d}t}U(t,t_i) = H(t)U(t,t_i), \tag{6.6}$$

where $H(t) = H_0(t) + H_I(t)$ is the full Hamiltonian, which consists of free + interaction parts and may be time dependent. The vacuum at any time t is defined by $H_0(t)\big|\,0_t\big\rangle$, which indicates that the true vacuum may be time dependent for time dependent Hamiltonians. The formal solution to the above equation is

$$U(t_f, t_i) = \exp\left(-i\int_{t_i}^{t_f} dt'\, H(t')\right), \quad t_i < t_f. \tag{6.7}$$

Evolution backwards in time, from t_f to t_i is given by

$$U(t_i, t_f) = \exp\left(-i\int_{t_f}^{t_i} dt'\, H(t')\right) = U(t_f, t_i)^\dagger, \tag{6.8}$$

[1] We will refer to this method as the closed time path (CPT) method since it describes best what is happening mathematically, but in the literature the notions Schwinger-Keldysh or *in-in* formalism are exchangeably used.

since the Hamiltonian is a hermitian operator.
We have seen that in general our *in*- and *out*-vacua are different states. In the Schrödinger picture, they evolve as

$$\big|\, in_f \big\rangle = U(t_f, t_i) \big|\, in_i \big\rangle \quad \text{and} \quad \big|\, out_f \big\rangle = U(t_f, t_i) \big|\, out_i \big\rangle, \tag{6.9}$$

where we will use the subscript notation to specify the time in the Schrödinger/interaction picture and omit the subscript for time independent states in the Heisenberg picture. The Heisenberg states are defined such that $\big|\, in \big\rangle = \big|\, in_i \big\rangle$, $\big|\, out \big\rangle = \big|\, out_f \big\rangle$ and $H(t_i)\big|\, in \big\rangle$, $H(t_f)\big|\, out \big\rangle$ give the minimum energy. The expectation value of some observable, represented by the operator $O(t)$ with respect to an arbitrary initial state, represented by the density matrix ρ_i at time t_i, at some other time t is given by

$$\langle O(t) \rangle_\rho = \frac{\mathrm{tr}\left(U(t,t_i)^\dagger O(t) U(t,t_i) \rho_i \right)}{\mathrm{tr}(\rho_i)}. \tag{6.10}$$

The density matrix reflects the vacuum choice, e.g. for a *in*-*in*-expectation value one can choose $\rho_i = \big|\, in_i \big\rangle \big\langle in_i \,\big|$ and hence

$$\langle O(t) \rangle_{in-in} = \left\langle in_i \,\middle|\, U(t,t_i)^\dagger O(t) U(t,t_i) \,\middle|\, in_i \right\rangle. \tag{6.11}$$

assuming $\big\langle in_i \,\big|\, in_i \big\rangle = 1$ and similarly for the *out*-*out*-expectation value. From eq. 4.2 we observe that in the path integral formalism the overlap of vacuum states can be represented as [1, 15, 23]

$$\langle out \,|\, U(t_f,t_i) \,|\, in \rangle = \big\langle out_* \,\big|\, in_* \big\rangle = \int_{\phi(x_i)=\phi(t_i,\boldsymbol{x})}^{\phi(x_f)=\phi(t_f,\boldsymbol{x})} \mathcal{D}\phi \; e^{iS[\phi]}, \tag{6.12}$$

for some field ϕ with general action $S[\phi]$, where subscript in $\big\langle out_* \,\big|\, in_* \big\rangle$ is notation for the overlap at an arbitrarily chosen time. As we have seen $\big\langle out_f \,\big|\, in_i \big\rangle \neq 1$ in general curved spacetimes, due to particle production out of gravitational energy. Field expectation values are obtained by introducing an artificial source term,

$$Z[J, g_{\mu\nu}] = \langle out \,|\, U_J(t_f,t_i) \,|\, in \rangle = \int_{\phi(x_i)}^{\phi(x_f)} \mathcal{D}\phi \; e^{iS[\phi, g_{\mu\nu}] + i\int d^dx \sqrt{|g|} J\phi}, \tag{6.13}$$

and then differentiating the generating functional with respect to it

$$\begin{aligned} &\langle out \,|\, \mathcal{T}\{\phi(x_1)\ldots\phi(x_n)\} \,|\, in \rangle = \\ &= \frac{1}{\sqrt{|g(x_1)|}} \frac{\delta}{\delta i J(x_1)} \cdots \frac{1}{\sqrt{|g(x_n)|}} \frac{\delta}{\delta i J(x_n)} \ln Z[J, g_{\mu\nu}] \bigg|_{J=0}, \end{aligned} \tag{6.14}$$

where $\mathcal{T}\{.\}$ denotes time ordering. The source term in the logarithm introduces a factor $1/Z[J, g_{\mu\nu}]$ to cancel vacuum diagrams. Time ordering appears naturally in the path integral formalism (see app. E).
Since we included a source term in the action the Hamiltonian changes and the evolution operator must also be adapted accordingly,

$$U_J(t_f,t_i) = \exp\left(-i \int_{t_f}^{t_i} dt' \, H(t') + i \int d^dx \sqrt{|g|} J\phi \right). \tag{6.15}$$

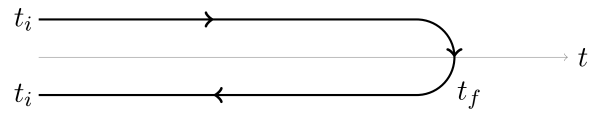


Figure 6.1: The closed time path contour, where we integrate from some initial time t_i up to a specific time t_f and then back to t_i.

Hence, for the *in-out* expectation value in eq. 6.13 we evolve the *in*-vacuum under the influence of the background curvature *and* a single source J and then compare this to the *out*-vacuum. In the CTP formalism one considers the influence of two sources $J^\pm$ on the *in*-vacuum and then compares the result at some arbitrary time $t_f > t_i$ in the future. In a mathematical language, we want to define a quantity $Z[J^+, J^-, g_{\mu\nu}]$, which corresponds to [35]

$$Z[J^+, J^-, g_{\mu\nu}] = {}_{J^-}\left\langle in \,|\, in \right\rangle_{J^+} = \left\langle in \,\middle|\, U_{J^-}(t,t_i)^\dagger U_{J^+}(t,t_i) \,\middle|\, in \right\rangle . \tag{6.16}$$

With this in mind we can introduce two artificial fields $\phi^\pm$, corresponding to the two sources $J^\pm$ and construct[2] [31, 32, 35, 36]

$$\begin{aligned} &Z[J^+, J^-, g_{\mu\nu}] = \\ &= \int_{\phi(x_i)}^{\phi(x_f)} \mathcal{D}\phi^+ \int_{\phi(x_f)}^{\phi(x_i)} \mathcal{D}\phi^- \\ &\quad \times \exp\left(iS[\phi^+, g_{\mu\nu}] + i\int d^dx \sqrt{|g|} J^+\phi^+ - iS[\phi^-, g_{\mu\nu}] - i\int d^dx\sqrt{|g|}J^-\phi^- \right), \end{aligned} \tag{6.17}$$

Diagrammatically the path integral can be thought of as integrating along the contour shown in fig. 6.1, giving it the name *closed time path* formalism. The quantities ϕ^+ and J^+ live on the upper branch going from $t_i \to t_f$ and ϕ^- and J^- on the lower one going back from $t_f \to t_i$. But we introduced these quantities only as mathematical tools and they still represent the physical field ϕ. Therefore, at the point t_f the fields and their derivatives must match

$$\phi^+(x_f) = \phi^-(x_f) \quad \text{and} \quad \partial_t\phi^+(x)|_{x_f} = \partial_t\phi^-(x)|_{x_f}. \tag{6.18}$$

In the end, only the physical field is relevant and we have to set $\phi^\pm$ to ϕ.
If we evolve the system forward and then backward in time under the influence of the same source, we expect nothing to happen and our normalisation to be reproduced. Indeed, setting the sources equal in eq. 6.17 we see

$$Z[J, J, g_{\mu\nu}] = \left\langle in \,|\, in \right\rangle = 1. \tag{6.19}$$

Let us now turn to correlation functions. Since we now have two sources to take a variation with respect to, we will have to investigate the differences. Varying with respect to J^+ we get the usual time ordered correlation function [31],

$$\begin{aligned} &\langle in \,|\, \mathcal{T}\{\phi(x_1)\ldots\phi(x_n)\} \,|\, in\rangle = \\ &= \frac{1}{\sqrt{|g(x_1)|}} \frac{\delta}{\delta i J^+(x_1)} \cdots \frac{1}{\sqrt{|g(x_n)|}} \frac{\delta}{\delta i J^+(x_n)} Z[J^+, J^-, g_{\mu\nu}] \Big|_{J^\pm = 0} . \end{aligned} \tag{6.20}$$

[2]This expression is generally only valid for pure states and a discussion of the generalisation can be found in [36], but since we want to only deal with *in-in* pure states this simplified approach will be sufficient for our analysis.

When we vary with respect to J^- we have to keep in mind that the time direction is reversed. We are moving form a later time t_f to an earlier one t_i. This will be reflected in the correlation function and we hence obtain anti-time ordering [31],

$$\begin{aligned} &\langle in \,|\, \bar{\mathcal{T}}\{\phi(x_1)\ldots\phi(x_n)\} \,|\, in\rangle = \\ &= \frac{1}{\sqrt{|g(x_1)|}}\frac{\delta}{\delta i J^-(x_1)}\cdots\frac{1}{\sqrt{|g(x_n)|}}\frac{\delta}{\delta i J^-(x_n)} Z[J^+,J^-,g_{\mu\nu}]\Big|_{J^\pm=0}, \end{aligned} \tag{6.21}$$

where $\bar{\mathcal{T}}\{.\}$ denotes anti-time ordering. Lastly, we can combine variation with respect to J^+ and J^-. In this case, note that the fields ϕ^- exist only on the later half of the time contour and hence will always appear ordered after the ϕ^+ fields in the correlation function. Therefore,

$$\begin{aligned} &\langle in \,|\, \bar{\mathcal{T}}\{\phi(x_1)\ldots\phi(x_n)\}\mathcal{T}\{\phi(x_{n+1})\ldots\phi(x_m)\} \,|\, in\rangle = \\ &= \frac{1}{\sqrt{|g(x_1)|}}\frac{\delta}{\delta i J^-(x_1)}\cdots\frac{1}{\sqrt{|g(x_n)|}}\frac{\delta}{\delta i J^-(x_n)} \\ &\times \frac{1}{\sqrt{|g(x_{n+1})|}}\frac{\delta}{\delta i J^+(x_{n+1})}\cdots\frac{1}{\sqrt{|g(x_m)|}}\frac{\delta}{\delta i J^+(x_m)} Z[J^+,J^-,g_{\mu\nu}]\Big|_{J^\pm=0}. \end{aligned} \tag{6.22}$$

There are no vacuum diagrams to cancel and consequently there is no need to take the logarithm of our generating functional $Z[J^+, J^-, g_{\mu\nu}]$ to obtain the correlation functions. This is reflected in the condition in eq. 6.19.
In the free field theory the integral in eq. 6.17 is Gaussian and can be integrated [23]. In this case we obtain [31, 35]

$$\begin{aligned} Z_0[J^+,J^-,g_{\mu\nu}] = \exp\Bigg[-\frac{i}{2}\int & d^dx d^dy \sqrt{|g(x)|}\sqrt{|g(y)|} \\ & \big(J^+(x)G_0^{++}(x,y)J^+(y) + J^-(x)G_0^{--}(x,y)J^-(y) \\ & - J^+(x)G_0^{+-}(x,y)J^-(y) - J^-(x)G_0^{-+}(x,y)J^+(y)\big)\Bigg], \end{aligned} \tag{6.23}$$

where the subscripts on the generating functional and the Green functions indicates the loop level – here we are considering a free theory. From the property in eq. 6.19 we find that the Green functions must obey

$$G_0^{++} + G_0^{--} - G_0^{+-} - G_0^{-+} = 0. \tag{6.24}$$

The Green functions are the solutions to the integral equations [31]

$$\begin{aligned} &\int d^dy \sqrt{|g(y)|}K(x,y)G_0^{\pm\pm}(y,z) = \pm\frac{1}{\sqrt{|g(x)|}}\delta^d(x-z), \\ &\int d^dy \sqrt{|g(y)|}K(x,y)G_0^{\pm\mp}(y,z) = 0, \end{aligned} \tag{6.25}$$

where $K(x,y)$ is the differential operator giving the equation of motion. At the moment we keep $K(x,y)$ general, but later we will specify to the Klein-Gordon operator. Comparing to the known propagators of sec. 3, one finds that all four of the above propagators can be expressed as [8, 31, 32]

$$\begin{aligned} iG_0^{++}(x,y) &= \big\langle in \,\big|\, \mathcal{T}_C\{\phi^+(x)\phi^+(y)\} \,\big|\, in \big\rangle = \langle in \,|\, \mathcal{T}\{\phi(x)\phi(y)\} \,|\, in \rangle \\ iG_0^{--}(x,y) &= \big\langle in \,\big|\, \mathcal{T}_C\{\phi^-(x)\phi^-(y)\} \,\big|\, in \big\rangle = \big\langle in \,\big|\, \bar{\mathcal{T}}\{\phi(x)\phi(y)\} \,\big|\, in \big\rangle \\ iG_0^{-+}(x,y) &= \big\langle in \,\big|\, \mathcal{T}_C\{\phi^-(x)\phi^+(y)\} \,\big|\, in \big\rangle = \langle in \,|\, \phi(x)\phi(y) \,|\, in \rangle \\ iG_0^{+-}(x,y) &= \big\langle in \,\big|\, \mathcal{T}_C\{\phi^+(x)\phi^-(y)\} \,\big|\, in \big\rangle = \langle in \,|\, \phi(y)\phi(x) \,|\, in \rangle \end{aligned} \tag{6.26}$$

where $\mathcal{T}_C\{.\}$ corresponds to time ordering along the contour in fig. 6.1. The first two Green functions are symmetric in their arguments,

$$iG_0^{\pm\pm}(x,y) = iG_0^{\pm\pm}(y,x), \tag{6.27}$$

due to (anti-)time ordering. The second two Green functions obey [35]

$$iG_0^{\pm\mp}(x,y) = iG_0^{\mp\pm}(y,x). \tag{6.28}$$

We can reduce the number of propagators by one if we preform a rotation in the fields [8, 31]. We can define a new basis of fields through the superposition

$$\phi_c = \frac{1}{\sqrt{2}}\left(\phi^+ + \phi^-\right) \quad \text{and} \quad \phi_q = \frac{1}{\sqrt{2}}\left(\phi^+ - \phi^-\right), \tag{6.29}$$

with the subscript c standing for "classical" and q for "quantum", the notation stemming from condensed matter theory [8]. This rotation is referred to as *Keldysh rotation* and the fields ϕ_c and ϕ_q are known as *Keldysh basis* [31]. The tree level generating functional, eq. 6.23 can be written as [31]

$$\begin{aligned} &Z_0[J_c, J_q, g_{\mu\nu}] = \\ &= \exp\Bigg[-\frac{i}{2}\int d^dx\sqrt{|g(x)|}d^dy\sqrt{|g(y)|} \\ &\quad \times \Big(J_q(x)G_0^K(x,y)J_q(y) + J_q(x)G_0^R(x,y)J_c(y) + J_c(x)G_0^A(x,y)J_q(y)\Big)\Bigg], \end{aligned} \tag{6.30}$$

where $J_c = \frac{1}{\sqrt{2}}\left(J^+ + J^-\right)$ and $J_q = \frac{1}{\sqrt{2}}\left(J^+ - J^-\right)$. These new correlation functions can easily be related to the ones in eq. 6.26 [8, 31, 32],

$$\begin{aligned} iG_0^K(x,y) &:= \langle in \,|\, \phi_c(x)\phi_c(y) \,|\, in \rangle \\ &= iG^{++}(x,y) + iG^{--}(x,y) = \langle in \,|\, \{\phi(x), \phi(y)\} \,|\, in \rangle \\ iG_0^R(x,y) &:= \langle in \,|\, \phi_c(x)\phi_q(y) \,|\, in \rangle \\ &= iG^{-+}(x,y) - iG^{--}(x,y) = \Theta(t_x - t_y)\, \langle in \,|\, [\phi(x), \phi(y)] \,|\, in \rangle \\ iG_0^A(x,y) &:= \langle in \,|\, \phi_q(x)\phi_c(y) \,|\, in \rangle \\ &= iG^{+-}(x,y) - iG^{--}(x,y) = -\Theta(t_y - t_x)\, \langle in \,|\, [\phi(x), \phi(y)] \,|\, in \rangle \\ &\quad \langle in \,|\, \phi_q(x)\phi_q(y) \,|\, in \rangle = 0 \end{aligned} \tag{6.31}$$

where we see that the number of propagators is reduced by one as the q-q-correlation function is zero. These propagators can be represented diagrammatically as in fig. 6.2. $iG_0^K(x,y)$ is

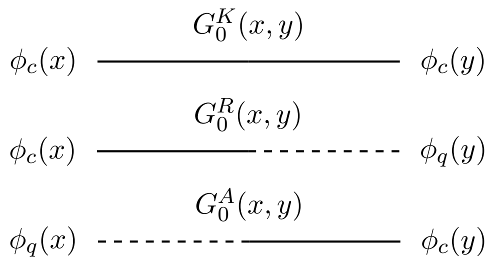


Figure 6.2: Diagrammatic representation of the Keldysh, retarded and advanced propagators respectively. The classical fields ϕ_c are represented by solid lines and the quantum fields ϕ_q are represented by dashed lines.

referred to as the *Keldysh* propagator. As found in sec. 3.7 (see eq. 3.87) for the free theory, the Keldysh propagator is invariant under $O(1,d)$. Additionally, it depends on the occupation number of the chosen Fock space or equivalently on the chosen vacuum and therefore on the background evolution [8, 9, 31].
The other two propagators $iG_0^R(x,y)$ and $iG_0^A(x,y)$ are referred to as *retarded* and *advanced propagators*, respectively. These propagators are independent of the chosen Fock space or vacuum as the commutator $[\phi(x),\phi(y)]$ is only a numerical quantity at tree level, as we have also found in sec. 3.7 (see eq. 3.85) [8, 9, 31].
For the free theory, we luckily already calculated the above propagators in previous sections. Since we have seen that the BD (or euclidean) vacuum corresponds to a good choice of *in*-vacuum, we can reuse our results based on 3.55. The tree level Keldysh propagator is just

$$iG_0^K(x,y) = G_+(Z_\epsilon(x,y)) + G_+(Z_{-\epsilon}(x,y)). \tag{6.32}$$

The retarded and advanced propagators can also be written in terms of the BD propagators as

$$\begin{aligned} iG_0^R(x,y) &= \Theta(t_x - t_y)\left(G_+(Z(x,y) - i\epsilon) - G_+(Z(x,y) + i\epsilon)\right), \\ iG_0^A(x,y) &= \Theta(t_y - t_x)\left(G_+(Z(x,y) - i\epsilon) - G_+(Z(x,y) + i\epsilon)\right) = iG_0^R(y,x), \end{aligned} \tag{6.33}$$

respectively.
In contrast to the Feynman diagrammatic technique the CTP formalism is completely causal, meaning that all loop contributions to the correlation functions depend only on the causal past of their arguments [8, 37].
Let us now move to partial momentum space by taking the spatial Fourier transform,

$$\mathcal{G}(k;\eta_x,\eta_y) = \int \text{đ}^{d-1}k\, e^{i\boldsymbol{k}.(\boldsymbol{x}-\boldsymbol{y})}\mathcal{G}(x,y), \tag{6.34}$$

of our propagators. To simplify the notation slightly, let us write our euclidean mode expansion of eq. 3.39 as

$$\phi(x) = \frac{1}{\sqrt{H}}(H\eta)^{\frac{d-1}{2}} \int \text{đ}^{d-1}k \left(a_{\boldsymbol{k}} h(k\eta)^* e^{i\boldsymbol{k}.\boldsymbol{x}} + a_{\boldsymbol{k}}^\dagger h(k\eta) e^{-i\boldsymbol{k}.\boldsymbol{x}}\right). \tag{6.35}$$

Substituting this mode expansion into eqs. 6.31, the free field propagators can be expressed in terms of the mode functions as

$$
\begin{aligned}
iG_0^K(k;\eta_x,\eta_y) &= \langle in \,|\, \{\phi(x),\phi(y)\} \,|\, in\rangle \\
&= \frac{1}{H}\left(H^2\eta_x\eta_y\right)^{\frac{d-1}{2}}\left(h(k\eta_x)^* h(k\eta_y) + h(k\eta_x)h(k\eta_y)^*\right) \\
iG_0^R(k;\eta_x,\eta_y) &= \Theta(\eta_x-\eta_y)\,\langle in \,|\, [\phi(x),\phi(y)] \,|\, in\rangle \\
&= \Theta(\eta_x-\eta_y)\frac{1}{H}\left(H^2\eta_x\eta_y\right)^{\frac{d-1}{2}}\left(h(k\eta_x)^* h(k\eta_y) - h(k\eta_x)h(k\eta_y)^*\right) \\
iG_0^A(k;\eta_x,\eta_y) &= -\Theta(\eta_y-\eta_x)\,\langle in \,|\, [\phi(x),\phi(y)] \,|\, in\rangle \\
&= -\Theta(\eta_y-\eta_x)\frac{1}{H}\left(H^2\eta_x\eta_y\right)^{\frac{d-1}{2}}\left(h(k\eta_x)^* h(k\eta_y) - h(k\eta_x)h(k\eta_y)^*\right).
\end{aligned}
\tag{6.36}
$$

For a general vacuum or excited state, where the vacuum is not annihilated by $a_{\boldsymbol{k}}$, the Keldysh propagator has the form

$$
\begin{aligned}
iG^K(k;\eta_x,\eta_y) = &\left(h(k\eta_x)^* h(k\eta_y) + h(k\eta_x)h(k\eta_y)^*\right)\left(1+2\langle a_{\boldsymbol{k}}^\dagger a_{\boldsymbol{k}}\rangle\right) \\
&+ 2h(k\eta_x)h(k\eta_y)\langle a_{\boldsymbol{k}}^\dagger a_{-\boldsymbol{k}}^\dagger\rangle + 2h(k\eta_x)^* h(k\eta_y)^*\langle a_{\boldsymbol{k}} a_{-\boldsymbol{k}}\rangle,
\end{aligned}
\tag{6.37}
$$

which again shows that the Keldysh propagator depends on the chosen Fock space. In quantum mechanics $\langle a_{\boldsymbol{k}}^\dagger a_{\boldsymbol{k}}\rangle$ is the usual number operator which gives information about the occupation number of the system per comoving volume [8]. $\langle a_{\boldsymbol{k}} a_{-\boldsymbol{k}}\rangle$ and its conjugate counterpart are known as the anomalous quantum averages, which can be interpreted as signalling the strength of backreaction on the background metric [8]. Although, one has to be very careful with adapting these interpretations for a dynamic background since we know that the particle concept is flawed. Therefore we will avoid these interpretations generally, denote

$$
n_k = 1 + 2\langle a_{\boldsymbol{k}}^\dagger a_{\boldsymbol{k}}\rangle \quad \text{and} \quad \kappa_k = \langle a_{\boldsymbol{k}} a_{-\boldsymbol{k}}\rangle \tag{6.38}
$$

and only use the notation as a mathematical tool. But the general Fock state dependence of the Keldysh propagator is very useful. Due to its connection to the occupation number its change can tell us about the strength of backreaction. In addition, we have seen that it is invariant under the full de Sitter group.
Due to our interested in the contribution of loop diagrams, their magnitude and therefore the amount of backreaction pn the spacetime geometry we can expect, we must consider self-interacting fields in the CTP formalism. Let us consider the action of an interacting $\lambda\phi^4$ theory

$$
S[\phi,g_{\mu\nu}] = \frac{1}{2}\int_i^f d^dx\sqrt{|g(x)|}\left[g^{\mu\nu}\nabla_\mu\phi(x)\nabla_\nu\phi(x) - M^2\phi^2(x) - \frac{\lambda}{4!}\phi^4(x)\right], \tag{6.39}
$$

where we have defined $M^2 = m^2 - \xi R$ as the effective mass. Preforming the above change of variables in the field,

$$
\begin{aligned}
\mathcal{S}[\phi_c,\phi_q,g_{\mu\nu}] &= S[\phi^+,g_{\mu\nu}] - S[\phi^-,g_{\mu\nu}] \\
&= \int_i^f d^dx\sqrt{|g|}\left[g^{\mu\nu}\nabla_\mu\phi_c\nabla_\nu\phi_q - M^2\phi_c\phi_q - \frac{\lambda}{4!}\phi_c\phi_q\left(\phi_c^2+\phi_q^2\right)\right].
\end{aligned}
\tag{6.40}
$$

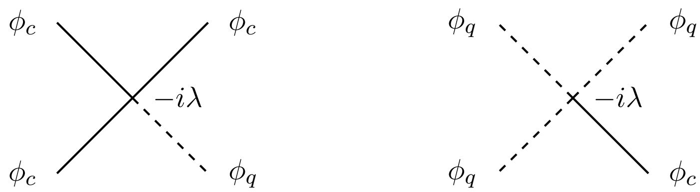


Figure 6.3: Diagrammatic representation of the four point interactions. The classical fields ϕ_c are represented by solid lines and the quantum fields ϕ_q are represented by dashed lines.

We see that for a purely classical field configuration, where $\phi_q = 0$ the action is zero [31, 33],

$$\mathcal{S}[\phi_c, \phi_q = 0, g_{\mu\nu}] = 0. \tag{6.41}$$

In our case this is also true for $\phi_c = 0$, but e.g. in $\lambda\phi^3$ theories,

$$\lambda\phi^3 = \frac{\lambda}{\sqrt{2}}\phi_q\left(3\phi_c^2 + \phi_q^2\right), \tag{6.42}$$

where this property indeed only holds for the purely classical configuration. Thinking in terms of the $\phi^{\pm}$ fields, when $\phi_q = 0$ we have $\phi^+ = \phi^-$ and the forward and backward propagation along the time contour cancel [33]. Thus it is only sensible that the action is zero as the forward and backward branch of the time contour exactly cancel.
Let us denote the interacting part of the action as

$$\mathcal{S}_I[\phi_c, \phi_q, g_{\mu\nu}] = \int d^dx\sqrt{|g|}\frac{\lambda}{4!}\left((\phi^+)^4 - (\phi^-)^4\right) = \int d^dx\sqrt{|g|}\frac{\lambda}{4!}\phi_c\phi_q\left(\phi_c^2 + \phi_q^2\right). \tag{6.43}$$

The possible vertex contributions are depicted in fig. 6.3. Treating the interaction as a perturbation, we can write the generating functional as

$$\begin{aligned} &Z[J^+, J^-, g_{\mu\nu}] = \\ &= \exp\left(i\mathcal{S}_I\left[\phi^+ \to \frac{1}{\sqrt{|g|}}\frac{\delta}{\delta iJ^+}, \phi^- \to \frac{1}{\sqrt{|g|}}\frac{\delta}{\delta iJ^-}, g_{\mu\nu}\right]\right) Z_0[J^+, J^-, g_{\mu\nu}]. \end{aligned} \tag{6.44}$$

The Feynman rules are [31]:

- For propagation between two field configurations $\phi^{(\pm)}(x)$ and $\phi^{(\pm)}(y)$ a propagator factor $G_0^{(\pm)(\pm)}(x, y)$ is introduced.
- Each vertex should be integrated over, giving a contribution of $-i\lambda\int d^dx\sqrt{|g(x)|}$.
- Since the $S[\phi^-, g_{\mu\nu}]$ appears with an additional minus sign, each such vertex gives an extra factor of -1.
- Lastly, we must include symmetry factors.

Alternatively we can express the generating functional in the Keldysh basis. Considering that

$$J^+\phi^+ - J^-\phi^- = J_c\phi_q + J_q\phi_c, \tag{6.45}$$

the generating functional becomes

$$Z[J_c, J_q, g_{\mu\nu}] = \exp\left(i\mathcal{S}_I\left[\phi_c \to \frac{1}{\sqrt{|g|}}\frac{\delta}{\delta iJ_q}, \phi_q \to \frac{1}{\sqrt{|g|}}\frac{\delta}{\delta iJ_c}, g_{\mu\nu}\right]\right) Z_0[J_c, J_q, g_{\mu\nu}], \tag{6.46}$$

where the fields in the interaction part of the action are replaced by variations with respect to the sources according to the above equation. For these fields, the Feynman rules are [31]:

- For propagation between two field configurations $\phi_{(c/q)}(x)$ and $\phi_{(c/q)}(y)$ the corresponding propagator factor $G_0^{K/R/A}(x,y)$ is introduced. In partial Fourier space, the propagators are represented by their corresponding counterparts $G_0^{K/R/A}(k;\eta_x,\eta_y)$.

- Each vertex should be integrated over, giving $-i\lambda\int d^dx\sqrt{|g(x)|}$. Alternatively, in partial Fourier space, we have to integrate over internal momenta and time, so that we have to integrate over $-i\lambda\int d\eta\int đ^{d-1}k$. (Note that for different combinations of fields, e.g. in eq. 6.42, corresponding numerical factors might appear and must be included.)

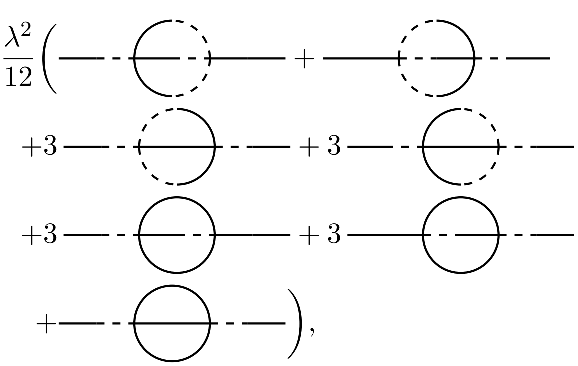
- Lastly, we must include symmetry factors.

We now want to utilize this analysis to find loop corrections to the Keldysh propagator in the Poincaré patch.

6.3 Loop correction in the expanding Poincaré patch

We are still most interested in the question of how the cosmological constant in de Sitter space might be screened due to particle production and interaction. Our discussion above leads to the conclusion that the Keldysh propagator is the most interesting propagator to consider, due to its dependence on the occupation number. Furthermore, since the UV limit only reproduces the flat space results, we are mostly interested in the IR nature of the loop corrections.
The lowest order loop corrections to the Keldysh propagator are [9, 31]

$$iG^K_{(2)}(x,y) = -\frac{\lambda^2}{12}$$



(6.47)

where the subscript labels only the two loop contributions. Using the above Feynman rules in partial Fourier space, we can represent the above diagrams in terms of propagators [9, 31],

$$\begin{aligned}
&iG^K_{(2)}(k;\eta_x,\eta_y) = \\
&= -\frac{\lambda^2}{12}\int_\infty^0\int_\infty^0 d\eta_u d\eta_v \int_0^\infty\int_0^\infty đ^{d-1}q_1 đ^{d-1}q_2 \\
&\times\big[-G^R_0(k;\eta_x,\eta_u)G^R_0(q_1;\eta_u,\eta_v)G^R_0(q_2;\eta_u,\eta_v)G^R_0(|\boldsymbol{k}-\boldsymbol{q}_1-\boldsymbol{q}_2|;\eta_u,\eta_v)G^K_0(k;\eta_v,\eta_y) \\
&\quad -G^K_0(k;\eta_x,\eta_u)G^A_0(q_1;\eta_u,\eta_v)G^A_0(q_2;\eta_u,\eta_v)G^A_0(|\boldsymbol{k}-\boldsymbol{q}_1-\boldsymbol{q}_2|;\eta_u,\eta_v)G^A_0(k;\eta_v,\eta_y) \\
&\quad -3G^R_0(k;\eta_x,\eta_u)G^A_0(q_1;\eta_u,\eta_v)G^A_0(q_2;\eta_u,\eta_v)G^K_0(|\boldsymbol{k}-\boldsymbol{q}_1-\boldsymbol{q}_2|;\eta_u,\eta_v)G^A_0(k;\eta_v,\eta_y) \\
&\quad -3G^R_0(k;\eta_x,\eta_u)G^R_0(q_1;\eta_u,\eta_v)G^R_0(q_2;\eta_u,\eta_v)G^K_0(|\boldsymbol{k}-\boldsymbol{q}_1-\boldsymbol{q}_2|;\eta_u,\eta_v)G^A_0(k;\eta_v,\eta_y) \\
&\quad +16G^R_0(k;\eta_x,\eta_u)G^K_0(q_1;\eta_u,\eta_v)G^K_0(q_2;\eta_u,\eta_v)G^R_0(|\boldsymbol{k}-\boldsymbol{q}_1-\boldsymbol{q}_2|;\eta_u,\eta_v)G^K_0(k;\eta_v,\eta_y) \\
&\quad +16G^K_0(k;\eta_x,\eta_u)G^K_0(q_1;\eta_u,\eta_v)G^K_0(q_2;\eta_u,\eta_v)G^A_0(|\boldsymbol{k}-\boldsymbol{q}_1-\boldsymbol{q}_2|;\eta_u,\eta_v)G^A_0(k;\eta_v,\eta_y) \\
&\quad +G^R_0(k;\eta_x,\eta_u)G^K_0(q_1;\eta_u,\eta_v)G^K_0(q_2;\eta_u,\eta_v)G^K_0(|\boldsymbol{k}-\boldsymbol{q}_1-\boldsymbol{q}_2|;\eta_u,\eta_v)G^A_0(k;\eta_v,\eta_y)\big].
\end{aligned} \tag{6.48}$$

Since we are only interested in the behaviour of the loop corrections in the IR limit, there is a number of approximations one can make to find the leading order contributions. This has been done in various publications [8, 9, 31], etc.. In these references mostly ϕ^3 interactions were discussed, since there the calculations are slightly easier and the fact that the theory is unstable is only of secondary concern. Additionally, one has to distinguish different mass ranges since the mode functions in eq. 3.39 behave differently depending on the magnitude of m/H. We will focus on the complementary series, where $n \in \mathbb{R}$. For a conformally coupled field in $d = 4$, $n = \sqrt{\frac{1}{4} - \frac{m^2}{H^2}}$. Therefore, for the following calculation we will assume $0 < n < 1/2$.
For the approximate evaluation of the expression in eq. 6.48, I found it simplest to follow [9] and [8] with additional input from [5] since these authors have done most work in the field up to this point. We will assume the above expression to be correctly UV regularised, so that we do not have to bother with UV divergences. But this is not really a problem in partial Fourier space, since we have full control over the magnitude of our momentum.
Let us start by writing eq. 6.48 in terms of mode functions as

$$\begin{aligned}
&iG^K_{(2)}(k;\eta_x,\eta_y) \approx \\
&\approx -\frac{i\lambda^2}{12H}(H\eta_x H\eta_y)^{\frac{d-1}{2}}\Big(h(k\eta_x)^* h(k\eta_y) n^{(2)}_k(\eta) + h(k\eta_x)^* h(k\eta_y)^* \kappa^{(2)}_k(\eta) + c.c\Big)
\end{aligned} \tag{6.49}$$

where in $n^{(2)}_k(\eta)$ and $\kappa^{(2)}_k(\eta)$ – representing the two loop contribution to n_k and κ_k – we have replaced the conformal times $\eta_{x,y}$ by the average time $\eta = \sqrt{\eta_x\eta_y} = \frac{1}{H}e^{-H(t_x+t_y)/2}$, which corresponds to the IR limit when taking both times to future infinity. This approximation takes care of the Θ-functions containing $\eta_{x,y}$ in the retarded and advanced propagators. The Θ-functions containing $\eta_{u,v}$ can all be brought to the same form by exchanging $\eta_u \leftrightarrow \eta_v$ in some loop contributions. Although, here we will neglect this theta function and fix the upper bound of $\eta_{u,v}$ to η. These expressions are

$$\begin{aligned}
n^{(2)}_k(\eta) \approx& \frac{1}{H}\int_\infty^\eta\int_\infty^\eta d\eta_u d\eta_y h(k\eta_u)h(k\eta_v)^* F_n(k,\eta_u,\eta_v) \\
\kappa^{(2)}_k(\eta) \approx& -\frac{1}{H}\int_\infty^\eta\int_\infty^\eta d\eta_u d\eta_v h(k\eta_u)h(k\eta_v) F_\kappa(k,\eta_u,\eta_v).
\end{aligned} \tag{6.50}$$

We have summarised the actual loop expressions in

$$\begin{aligned}F_n(k,\eta_u,\eta_v) =&\frac{1}{H^3}(H\eta_u H\eta_v)^{d-2}\int_0^\infty\int_0^\infty đ^{d-1}q_1 đ^{d-1}q_2\\ &\big[35h\left(\eta_v q_1\right)h\left(\eta_v q_2\right)h\left(\eta_v\left(q_1+q_2\right)\right)h^*\left(\eta_u q_1\right)h^*\left(\eta_u q_2\right)h^*\left(\eta_u\left(q_1+q_2\right)\right)\\ &+39h\left(\eta_u q_1\right)h\left(\eta_v q_2\right)h\left(\eta_v\left(q_1+q_2\right)\right)h^*\left(\eta_v q_1\right)h^*\left(\eta_u q_2\right)h^*\left(\eta_u\left(q_1+q_2\right)\right)\\ &+27h\left(\eta_v q_1\right)h\left(\eta_u q_2\right)h\left(\eta_v\left(q_1+q_2\right)\right)h^*\left(\eta_u q_1\right)h^*\left(\eta_v q_2\right)h^*\left(\eta_u\left(q_1+q_2\right)\right)\\ &+23h\left(\eta_u q_1\right)h\left(\eta_u q_2\right)h\left(\eta_v\left(q_1+q_2\right)\right)h^*\left(\eta_v q_1\right)h^*\left(\eta_v q_2\right)h^*\left(\eta_u\left(q_1+q_2\right)\right)\\ &-37h\left(\eta_v q_1\right)h\left(\eta_v q_2\right)h\left(\eta_u\left(q_1+q_2\right)\right)h^*\left(\eta_u q_1\right)h^*\left(\eta_u q_2\right)h^*\left(\eta_v\left(q_1+q_2\right)\right)\\ &-41h\left(\eta_u q_1\right)h\left(\eta_v q_2\right)h\left(\eta_u\left(q_1+q_2\right)\right)h^*\left(\eta_v q_1\right)h^*\left(\eta_u q_2\right)h^*\left(\eta_v\left(q_1+q_2\right)\right)\\ &-29h\left(\eta_v q_1\right)h\left(\eta_u q_2\right)h\left(\eta_u\left(q_1+q_2\right)\right)h^*\left(\eta_u q_1\right)h^*\left(\eta_v q_2\right)h^*\left(\eta_v\left(q_1+q_2\right)\right)\\ &-25h\left(\eta_u q_1\right)h\left(\eta_u q_2\right)h\left(\eta_u\left(q_1+q_2\right)\right)h^*\left(\eta_v q_1\right)h^*\left(\eta_v q_2\right)h^*\left(\eta_v\left(q_1+q_2\right)\right)\big],\end{aligned} \tag{6.51}$$

$$\begin{aligned}F_\kappa(k,\eta_u,\eta_v) =&\frac{1}{H^3}(H\eta_u H\eta_v)^{d-2}\int_0^\infty\int_0^\infty đ^{d-1}q_1 đ^{d-1}q_2\\ &\big[-7h\left(\eta_v q_1\right)h\left(\eta_v q_2\right)h\left(\eta_v\left(q_1+q_2\right)\right)h^*\left(\eta_u q_1\right)h^*\left(\eta_u q_2\right)h^*\left(\eta_u\left(q_1+q_2\right)\right)\\ &-7h\left(\eta_u q_1\right)h\left(\eta_v q_2\right)h\left(\eta_v\left(q_1+q_2\right)\right)h^*\left(\eta_v q_1\right)h^*\left(\eta_u q_2\right)h^*\left(\eta_u\left(q_1+q_2\right)\right)\\ &+5h\left(\eta_v q_1\right)h\left(\eta_u q_2\right)h\left(\eta_v\left(q_1+q_2\right)\right)h^*\left(\eta_u q_1\right)h^*\left(\eta_v q_2\right)h^*\left(\eta_u\left(q_1+q_2\right)\right)\\ &+5h\left(\eta_u q_1\right)h\left(\eta_u q_2\right)h\left(\eta_v\left(q_1+q_2\right)\right)h^*\left(\eta_v q_1\right)h^*\left(\eta_v q_2\right)h^*\left(\eta_u\left(q_1+q_2\right)\right)\\ &+5h\left(\eta_v q_1\right)h\left(\eta_v q_2\right)h\left(\eta_u\left(q_1+q_2\right)\right)h^*\left(\eta_u q_1\right)h^*\left(\eta_u q_2\right)h^*\left(\eta_v\left(q_1+q_2\right)\right)\\ &+5h\left(\eta_u q_1\right)h\left(\eta_v q_2\right)h\left(\eta_u\left(q_1+q_2\right)\right)h^*\left(\eta_v q_1\right)h^*\left(\eta_u q_2\right)h^*\left(\eta_v\left(q_1+q_2\right)\right)\\ &-7h\left(\eta_v q_1\right)h\left(\eta_u q_2\right)h\left(\eta_u\left(q_1+q_2\right)\right)h^*\left(\eta_u q_1\right)h^*\left(\eta_v q_2\right)h^*\left(\eta_v\left(q_1+q_2\right)\right)\\ &-7h\left(\eta_u q_1\right)h\left(\eta_u q_2\right)h\left(\eta_u\left(q_1+q_2\right)\right)h^*\left(\eta_v q_1\right)h^*\left(\eta_v q_2\right)h^*\left(\eta_v\left(q_1+q_2\right)\right)\big],\end{aligned} \tag{6.52}$$

where F_n and F_κ are essentially the same, up to the respective numerical coefficients of the individual terms.
Recall that in the IR limit, the the BD modes behave as in eq. 6.1. Since we set the range on n to $0 < n < 1/2$ all the Γ-functions in the above expressions are purely real. Due to the symmetry in complex conjugation in eq. 6.49, there will be some simplifications.
First, we start by investigating the behaviour of $F_n(k,\eta_u,\eta_v)$ and $F_\kappa(k,\eta_u,\eta_v)$. Since all the terms in the expressions are the same up to different combinations of Hankel functions and their corresponding conjugates, we can pick one particular term and use it as a representative,

$$\begin{aligned}F(k,\eta_u,\eta_v) =&\frac{1}{H^3}(H\eta_u H\eta_v)^{d-2}\int_0^\infty\int_0^\infty đ^{d-1}q_1 đ^{d-1}q_2\\ &\times h(q_1\eta_u)^* h(q_1\eta_v) h(q_2\eta_u)^* h(q_2\eta_v)\\ &\times h(|\boldsymbol{k}-\boldsymbol{q}_1-\boldsymbol{q}_2|\,\eta_u)^* h(|\boldsymbol{k}-\boldsymbol{q}_1-\boldsymbol{q}_2|\,\eta_v)\end{aligned} \tag{6.53}$$

All other terms will behave similarly. We can split the momentum integrals as

$$\int_0^\infty\int_0^\infty đ^{d-1}q_1 đ^{d-1}q_2 = \left(\int_{q_1<k}+\int_{q_1>k}\right)\left(\int_{q_2<k}+\int_{q_2>k}\right)đ^{d-1}q_1 đ^{d-1}q_2 \tag{6.54}$$

which then gives four regions of integration

1. $q_1 < k$, $q_2 < k$
2. $q_1 > k$, $q_2 > k$
3. $q_1 < k$, $q_2 > k$
4. $q_1 > k$, $q_2 < k$.

The contribution of the first interval can be estimated by approximating $|\boldsymbol{k} - \boldsymbol{q}_1 - \boldsymbol{q}_2| \approx k$ and then taking the small k limit,

$$\begin{aligned} F_1(k, \eta_u, \eta_v) \approx & \frac{1}{H^3}(H\eta_u H\eta_v)^{d-2} h(k\eta_u)^* h(k\eta_v) \\ & \times \int_{q_1<k}\int_{q_2<k} \text{đ}^{d-1}q_1 \text{đ}^{d-1}q_2 h(q_1\eta_u)^* h(q_1\eta_v) h(q_2\eta_u)^* h(q_2\eta_v) \\ \sim & (\eta_u\eta_v)^{d-2-3n} k^{-2n} \int_{q_1<k}\int_{q_2<k} \text{đ}^{d-1}q_1 \text{đ}^{d-1}q_2 (q_1 q_2)^{-2n} \\ \sim & (\eta_u\eta_v)^{d-2-3n} k^{2(d-1-3n)}, \end{aligned} \tag{6.55}$$

which is well behaved in the IR limit for $d - 1 - 3n > 0$.
In the second interval range we can approximate $|\boldsymbol{k} - \boldsymbol{q}_1 - \boldsymbol{q}_2| \approx |\boldsymbol{q}_1 + \boldsymbol{q}_2|$, giving

$$\begin{aligned} F_2(k, \eta_u, \eta_v) \approx & \frac{1}{H^3}(H\eta_u H\eta_v)^{d-2} \int_{q_1>k}\int_{q_2>k} \text{đ}^{d-1}q_1 \text{đ}^{d-1}q_2 \\ & \times h(q_1\eta_u)^* h(q_1\eta_v) h(q_2\eta_u)^* h(q_2\eta_v) h(|\boldsymbol{q}_1 + \boldsymbol{q}_2|\,\eta_u)^* h(|\boldsymbol{q}_1 + \boldsymbol{q}_2|\,\eta_v), \end{aligned} \tag{6.56}$$

which in this approximation gets rid of the k dependence in $F_2(k, \eta_u, \eta_v) \approx F_2(\eta_u, \eta_v)$ when we take $k \to 0$.
The integral in the intervals 3. and 4. behave similarly. Approximating $|\boldsymbol{k} - \boldsymbol{q}_1 - \boldsymbol{q}_2| \approx q_2$,

$$\begin{aligned} F_3(k, \eta_u, \eta_v) \approx & F_4(k, \eta_u, \eta_v) \\ \sim & (\eta_u\eta_v)^{d-2-n} \int_{q_1<k}\int_{q_2>k} \text{đ}^{d-1}q_1 \; \text{đ}^{d-1}q_2 \\ & \times q_1^{-2n} h(q_2\eta_u)^* h(q_2\eta_v) h(q_2\eta_u)^* h(q_2\eta_v) \\ \sim & (\eta_u\eta_v)^{d-2-n} k^{d-1-2n} \int_{q_2>k} \text{đ}^{d-1}q_2 h(q_2\eta_u)^* h(q_2\eta_v) h(q_2\eta_u)^* h(q_2\eta_v), \end{aligned} \tag{6.57}$$

which is well behaved in the IR limit for $d - 1 - 2n > 0$.
Thus, we have the largest possible IR contribution from the region $q_{1,2} < k$ where $F_1 \propto k^{2(d-1-3n)}$. Additionally, as long as $d - 1 - 3n > 0$, $F(k, \eta_u, \eta_v)$ is well behaved and will not give large IR contributions.
Therefore, we can neglect k in comparison with $q_{1,2}$ in $F(k, \eta_u, \eta_v)$ and write

$$F_n(k, \eta_u, \eta_v) \approx F_\kappa(k, \eta_u, \eta_v) \approx -8F(k, \eta_u, \eta_v), \tag{6.58}$$

where the numerical factor comes from summing all the numerical coefficients in eq. 6.50. We want to extract the k dependence of eq. 6.49, and therefore would like to expand the mode functions for small argument. One can only do this by cutting off the $\eta_{u/v}$ integrals at n/k, which corresponds to neglecting the high comoving momenta with $k\eta_{u/v} \gg n$,

$$\int_\infty^\eta \int_\infty^\eta d\eta_u \; d\eta_v \to \int_{n/k}^\eta \int_{n/k}^\eta d\eta_u \; d\eta_v. \tag{6.59}$$

Otherwise, expanding the mode functions would not be valid at the lower ends of the conformal time integrals (corresponding to $\eta \to \infty$). This corresponds to neglecting high momentum UV behaviour, which we are not interested in here anyway.
Next, we preform the substitution $u = k\sqrt{\eta_u \eta_v}$ and $v = \sqrt{\eta_u/\eta_v}$. Our integral measure then becomes

$$\int_{n/k}^{\eta} \int_{n/k}^{\eta} d\eta_u \, d\eta_v \to -\frac{2}{k^2} \int_{n}^{k\eta} \int du \, dv \, \frac{u}{v}. \tag{6.60}$$

Additionally we shift our momenta as $q_{1,2} \to q_{1,2}/\eta_v = q_{1,2} kv/u$ so that

$$\begin{aligned} F(u,v) =& \frac{(Hv)^{2(d-1)}}{H^5(ku)^2} \int_0^\infty \int_0^\infty đ^{d-1} q_1 đ^{d-1} q_2 \\ & h(q_1 v^2)^* h(q_1) h(q_2 v^2)^* h(q_2) h(|\boldsymbol{q}_1 + \boldsymbol{q}_2| \, v^2)^* h(|\boldsymbol{q}_1 + \boldsymbol{q}_2|) \end{aligned} \tag{6.61}$$

As an intermediate summary, we have

$$\begin{aligned} iG^K_{(2)}(k;\eta_x,\eta_y) \approx & -\frac{i\lambda^2}{12H} (H\eta_x H\eta_y)^{\frac{d-1}{2}} \\ & \times \left(h(k\eta_x)^* h(k\eta_y) n_k^{(2)}(\eta) + h(k\eta_x)^* h(k\eta_y)^* \kappa_k^{(2)}(\eta) + c.c \right) \\ n_k^{(2)}(\eta) \approx & -\frac{16 H^{2(d-1)}}{H^3 k^4} \int_n^{k\eta} \int dudv \, \frac{v^{2d-3}}{u} h(uv) h(u/v)^* W(v) \\ \kappa_k^{(2)}(\eta) \approx & \frac{16 H^{2(d-1)}}{H^3 k^4} \int_n^{k\eta} \int dudv \, \frac{v^{2d-3}}{u} h(uv) h(u/v)^* W(v) \\ W(v) =& \frac{1}{H^3} \int_0^\infty \int_0^\infty đ^{d-1} q_1 đ^{d-1} q_2 \\ & \times h(q_1 v^2)^* h(q_1) h(q_2 v^2)^* h(q_2) h(|\boldsymbol{q}_1 + \boldsymbol{q}_2| \, v^2)^* h(|\boldsymbol{q}_1 + \boldsymbol{q}_2|) \end{aligned} \tag{6.62}$$

The small parameters in the above expression are k and u, since u also contains one power of k. One immediately sees that if we would expand $h(k\eta_{x,y})$, $h(uv)$ and $h(u/v)$ to smallest order as in eq. 6.1, the two terms would cancel. Hence we must go to a higher order of k.
We can expand $iG^K_{(2)}$ to lowest non-vanishing order in k and then preform the integral over u. Neglecting all orders which are finite in the limit $k \to 0$, we get

$$\begin{aligned} iG^K_{(2)}(k;\eta) =& i\lambda^2 \frac{4A_-^3 \, \mathrm{Re}\,(A_+)\, H^{2d-6}}{3} \frac{(H\eta)^{d-1}}{(k\eta)^{2n}} \log\left(\frac{k^2\eta^2}{n^2}\right) \\ & \times \int dv \, v^{2d-3} (-v^{-2n} + v^{2n}) \left(W(v) - W(v)^*\right), \end{aligned} \tag{6.63}$$

similar to the result obtained by others [5, 8, 9, 37], etc..
Hence the general form of the IR contributions to the Keldysh propagator up to second order loop effects is

$$iG^K_2(k;\eta) = i\frac{(H\eta)^{d-1}}{(k\eta)^{2n}} \left[A + \lambda^2 B \log\left(\frac{k^2\eta^2}{n^2}\right) \right], \tag{6.64}$$

where α, β are numerical coefficients. Therefore we get an additional, logarithmically diverging IR contribution to the Keldysh propagator as we take $k\eta \to 0$. This contribution is still de

Sitter invariant and hence a function of Z when we go back to position space. This can be seen by considering the parameter

$$Z \approx 1 + \frac{\eta^2 - \boldsymbol{x}^2}{2\eta^2} \tag{6.65}$$

In partial Fourier space Z is represented by $1/k\eta$ and therefore we see that the expression in eq. 6.64 is de Sitter invariant. Therefore this cannot be the origin of a sufficiently large backreaction on the spacetime and a breaking of the exact de Sitter universe.

6.4 De Sitter invariance at loop level

We have found that the second order loop corrections to the Keldysh propagator are exactly de Sitter invariant. This seems like a strange coincidence at first and one might ask if this is a general feature? Given the arguments in app. F, as long as we do not break de Sitter invariance, the contribution to the EMT will always be constant and we will not find growing IR contributions which inevitably cause backreaction on the spacetime geometry.
One can indeed argue that, if we restrict ourselves to the expanding Pinocaré patch, any loop correction will be de Sitter invariant, as long as we do not encounter IR divergences. These arguments are nicely summarised in [8] and we will reproduce them here.
An arbitrary order loop correction to the propagators will be of the form

$$\mathcal{G}_{(.)}(X,Y) = \int DU \int DV\, \mathcal{G}_0(Z_\epsilon(X,U))\Sigma_0(Z_\epsilon(U,V))\mathcal{G}_0(Z_\epsilon(V,Y)), \tag{6.66}$$

where $\mathcal{G}$ is a placeholder for any of the propagators $G_0^{K/R/A}$, X, Y, U, V are embedding coordinates, and DU, DV are the corresponding integral measures. $\Sigma_0(Z_\epsilon(U,V))$ is a combination of different propagators which determine the loop structure. The integral measures for the Poincaré patch are (see sec. 2.4)

$$DU = d^D U \;\delta(U_A U^A + H^{-2})\;\Theta(U^0 - U^D) = d^d x_U \sqrt{|g|} \tag{6.67}$$

and similarly for DV. Here the δ-function enforces the defining embedding property of de Sitter space from eq. 2.11 and the Θ-function restricts to the expanding Poincaré patch.
Consider now the small rotation of U^D into any other spatial coordinate

$$U^D \to U^D + \sigma U^1, \tag{6.68}$$

where U^1 is arbitrary here. We can expand the integral measure in σ, which gives the first order variation

$$\begin{aligned} \delta_\sigma DU &= d^D U\, \delta(U_A U^A + H^{-2})\, \delta(U^0 - U^D)\, \sigma U^1 \\ &= d(U^0 + U^D) d^{D-2} U\, \delta(U_A U^A + H^{-2})\, \sigma U^1. \end{aligned} \tag{6.69}$$

We already have established that the integrand in the loop corrections only depends on Z, which can be written as

$$\begin{aligned} Z(X,U) &= -H^2 \eta_{AB} X^A U^B \\ &= -H^2 \left[\frac{1}{2}(X^0 + X^D)(U^0 - U^D) + \frac{1}{2}(X^0 - X^D)(U^0 + U^D) - \delta_{ij} X^i U^j\right]. \end{aligned} \tag{6.70}$$

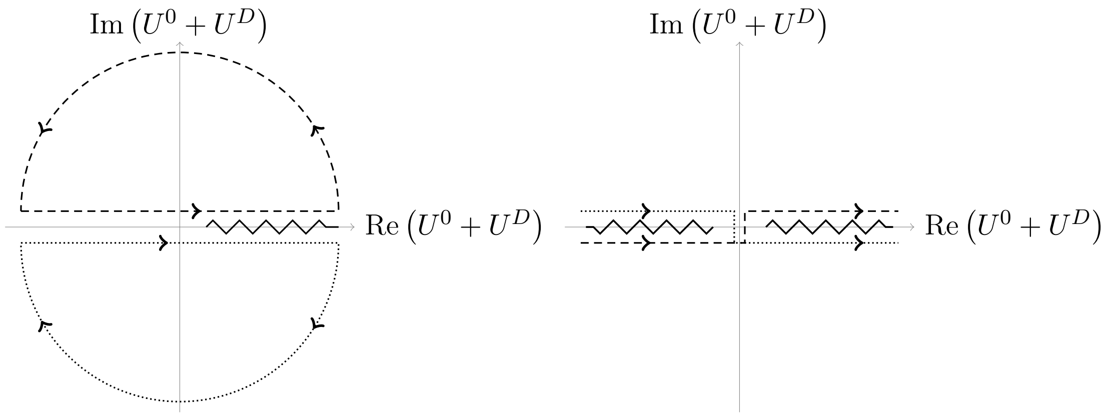


Figure 6.4: *Left*: The two possible integration contours of the $d(U^0 + U^D)$ integral for a BD propagator with branch cut from $1 \to \infty$ along the real axis. No poles are included in any of the contours.
Right: The integration contours of the $d(U^0 + U^D)$ integral for any other α propagator with an additional branch cut from $1 \to \infty$ along the real axis and also a different $i\epsilon$ prescription.

The first term in Z, containing $(U^0 - U^D)$, is zero in the variation of the measure due to the $\delta(U^0 - U^D)$ factor. We are in the expanding Poincaré patch and so $(X^0 - X^D) \geq 0$. Therefore $\mathcal{G}_0(Z(X, U|_{U^0 \to U^0 + U^D}))$ has the same analytic structure as $\mathcal{G}_0(Z(X, U))$.
Since the δ-distribution enforces $U^0 - U^D = 1/(H^2\eta_u) = 0$, η_u is pushed to past infinity, $\mathrm{sgn}(x, u) = +1$ and the $i\epsilon$-prescription of the propagators is fixed to

$$\mathcal{G}_0(X, U) = \mathcal{G}_0(Z(X, U) - i\epsilon), \tag{6.71}$$

and oppositely for $\mathcal{G}_0(Z_\epsilon(V, Y))$. Furthermore, for the BD two point function the integrand of $d(U^0 + U^D)$ is an analytic function in the complex $U^0 + U^D$ plane with a cut going from $1 \to \infty$ along the real axis, but slightly shifted due to the $i\epsilon$-prescription.
Since the propagators fall off quick enough for large $\left|U^0 + U^D\right|$, we can choose to close the integration contour with a large semi circle either in the upper or lower half $U^0 + U^D$ plane as sketched in fig. 6.4. Since $\mathcal{G}_0$ is analytic everywhere this integral will be zero by the residue theorem. Therefore the variation of the integral measure $\delta_\sigma DU$ (and also $\delta_\sigma DV$ by the same argument) vanishes when integrating the BD propagator.
For any other α-vacua, these arguments break down at the point where we try to close the integral contour. Here we have an additional branch cut from $-1 \to -\infty$, where our propagator also carries a different $i\epsilon$-prescription. Hence one cannot close the contour in a simple way.
Summarizing, we have found that as long as we consider only the Poincaré patch and BD type propagators, any loop correction will only be a function of the geodesic distance,

$$\mathcal{G}_{(.)}(X, Y) = \mathcal{G}_{(.)}(Z_\epsilon(X, Y)), \tag{6.72}$$

and we will never break de Sitter isometry. This agrees with our findings in the previous section. In return, this tells us that loop corrections will vanish in the coincidence limit and we will not get growing contributions to the EMT. On the other hand if one considers other vacua loop corrections will break the de Sitter isometry. Furthermore, in global de Sitter space or in the contracting Poincaré patch these arguments also break down, because one encounters IR divergences [8].

Chapter 7

Backreaction and graceful exit

So far we have learned that we must break de Sitter isometry to break out of the exponentially expanding phase and obtain a graceful exit. We have seen that this isometry is respected in the free theory and even in the loop corrections to the propagators. Thus, we can conclude that we need to break de Sitter isometry some other way. So far we have restricted ourselves to exact de Sitter space. We have not really allowed for a dynamic backreaction on the spacetime. This is what we want to try and address in this section.

A simple glance at the semi-classical Einstein equation (eq. 4.16)

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - g_{\mu\nu}\Lambda = 8\pi \langle T_{\mu\nu} \rangle, \tag{7.1}$$

shows that by construction we assume that the quantum effects onto the spacetime remain negligibly small, as de Sitter is a solution for $\langle T_{\mu\nu} \rangle = 0$. The Einstein equation assumes a fixed background so even if quantum effects would become large, we do not yet have physical understanding of the effects this would have. To investigate this behaviour, one must add some degree of freedom to our metric to take account for this backreaction. We will treat this degree of freedom as a scalar perturbation to the metric. If quantum effects indeed have a chance of becoming large, this will manifest itself in a breakdown of perturbation theory.
We will parametrize our metric by

$$g_{\mu\nu} = \frac{1}{\eta^2} \left(1 + s f(x)\right) \eta_{\mu\nu}, \quad s f(x) \ll 1, \tag{7.2}$$

where we have added a scalar degree of freedom, $f(x)$, to the original de Sitter metric of the expanding Poincaré patch and we have set $H = 1$ for simplicity. The parameter s will be our expansion parameter.
As always in general relativity, there is the question of coordinate dependent effects. We have seen in sec. 6.4 that the form of the loop corrections respects de Sitter invariance, given that we are working in the expanding Poincaré patch and do not encounter direct singularities. In other patches of the spacetime this is not so apparent or even not true. The same question one can ask of the result we obtained for our metric perturbation.
To obtain concrete and unquestionable results for our metric perturbation, one has to construct gauge invariant variables which remain unchanged under general coordinate transformations [1]. Due to gauge freedom we can choose the longitudinal gauge such that the chosen metric perturbation is indeed a gauge invariant variable, given that our EMT is diagonal [1].

The general approach will be as follows. We will linearise both the equation of motion of the scalar field and the trace of the Einstein equation. To zeroth order is the expansion we will reproduce our result from sec. 3 for the two point function. This result we can substitute into the Einstein equation to solve for $f(x)$. For simplicity we fix $d = 4$ and we will neglect the higher derivative terms from sec. 5.1. The Einstein then has the form

$$G^0_{\alpha\beta} + sG^1_{\alpha\beta} = \Lambda g_{\alpha\beta} + s\langle T_{\mu\nu}\rangle \tag{7.3}$$

where the superscripts label the order of s. The zeroth order $G^0_{\alpha\beta}$ will be free of the perturbation and $f(x)$ will only appear in $G^1_{\alpha\beta}$. To $\mathcal{O}(s^0)$ we have

$$G^0_{\alpha\beta} = \Lambda g_{\alpha\beta} \tag{7.4}$$

which just gives the standard de Sitter solution from 2.10 in $d = 4$,

$$\Lambda = 3. \tag{7.5}$$

The scalar field equation of motion from eq. 3.10 becomes

$$\begin{aligned}\Box_M \chi(x) + \Bigg[&\frac{(1+sf(x))^2}{\eta^2}\left(\frac{9}{4} - 12\xi - n^2\right) \\ &- \frac{(1-6\xi)}{(1+sf(x))}\left(s\Box_M f(x) - s\frac{2}{\eta}\partial_\eta f(x) + s\frac{2}{\eta^2}f(x) + \frac{2}{\eta^2}\right)\Bigg]\chi(x) = 0\end{aligned} \tag{7.6}$$

when written in terms of $n = \sqrt{\frac{9}{4} - \frac{m^2}{H^2} - 12\xi}$, which is slightly modified here as we have included the ξR coupling in the Lagrangian. We again see that a conformally coupled scalar field completely decouples from gravity as $\frac{9}{4} - 12\xi - n^2 = m^2$ is just the mass term. As usual, this equation resembles a harmonic oscillator with coordinate dependent frequency and can be expressed as

$$\Box_M \chi + \Omega[f(x), \eta]^2 \, \chi = 0, \tag{7.7}$$

where $\Omega[f(x), \eta]^2$ just symbolically hides all the mess appearing in 7.6. We can now expand in s to linearise the equation substitute the mode expansion from eq. 3.20. This then leads to

$$q_k''(\eta) + \frac{1}{\eta^2}\left(k^2\eta^2 - n^2 + \frac{1}{4}\right) q_k(\eta) = 0, \tag{7.8}$$

which is just eq. 3.21 for $d = 4$ written in terms of n. As we already know, the solutions to this equation are given by

$$q_k = \sqrt{\eta}\left(A_1 H_n^{(1)}(k\eta) + A_2 H_n^{(2)}(k\eta)\right), \tag{7.9}$$

with $H_n^{(1/2)}$ being Hankel functions.

At this point we must choose an appropriate vacuum for our expectation value, as otherwise we do not stand a chance of finding a solution to $f(x)$. For BD boundary conditions we then have

$$\chi_{\boldsymbol{k}} = \sqrt{\frac{\pi\eta}{2}}\left(a^\dagger_{\boldsymbol{k}} H_n^{(1)}(k\eta) + h.c.\right). \tag{7.10}$$

The two point function in partial momentum space becomes

$$\langle \chi_k \chi_k \rangle = \frac{\pi\eta}{2} \left| H_n^{(1)}(k\eta) \right|^2 . \tag{7.11}$$

Let us now turn to the Einstein equation. To $\mathcal{O}(s^1)$ the trace of the linearised Einstein equation becomes

$$\begin{aligned} &\Box_M f(x) - \frac{2}{\eta}\partial_\eta f(x) - \frac{4}{\eta^2} f(x) \\ &+ \frac{4\pi\eta}{3}(1-6\xi)\left[\eta^{\mu\nu}\langle \partial_\mu \chi(x) \partial_\nu \chi(x)\rangle + \frac{2}{\eta}\langle \chi(x)\partial_\eta \chi(x)\rangle\right] \\ &+ \frac{2\pi}{3}\left(39\xi + 4(1-3\xi)n^2 - 7\right)\langle \chi(x)\chi(x)\rangle = 0. \end{aligned} \tag{7.12}$$

We can now write this expression purely in terms of the two point function and derivatives thereof. To this end we can use the property

$$\begin{aligned} \eta^{\mu\nu}\langle \partial_\mu \chi(x) \partial_\nu \chi(x)\rangle &= \frac{1}{2}\Box_M \langle \chi(x)\chi(x)\rangle - \langle \chi(x)\Box_M \chi(x)\rangle \\ &= \frac{1}{2}\Box_M \langle \chi(x)\chi(x)\rangle + \Omega^0(\eta)^2 \langle \chi(x)\chi(x)\rangle, \end{aligned} \tag{7.13}$$

having made use of eq. 7.6. To zeroth order in the expansion

$$\Omega^0(\eta)^2 = \frac{1}{\eta^2}\left(k^2\eta^2 - n^2 + \frac{1}{4}\right) \tag{7.14}$$

is independent of $f(x)$. We can also transform our metric perturbation and two point function to partial Fourier space

$$f(x) = \int đ^{d-1}k f_k(\eta) e^{-i\boldsymbol{k}.\boldsymbol{x}} \quad \text{and} \quad \langle \chi(x)\chi(x)\rangle = \int đ^{d-1}k \langle \chi_k(\eta)\chi_k(\eta)\rangle e^{-i\boldsymbol{k}.\boldsymbol{x}}. \tag{7.15}$$

Then the Fourier transformed trace Einstein equation becomes

$$\begin{aligned} &f_k''(\eta) - \frac{2}{\eta} f_k'(\eta) + \frac{1}{\eta^2}\left(k^2\eta^2 - 4\right) f_k(\eta) \\ &- \frac{2\pi\eta^2}{3}(1-6\xi)\left(\langle \chi_k(\eta)\chi_k(\eta)\rangle'' - \frac{2}{\eta}\langle \chi_k(\eta)\chi_k(\eta)\rangle'\right) \\ &+ \frac{\pi}{3}\left(4n^2 + 72\xi - 13 - 2k^2\eta^2(1-6\xi)\right)\langle \chi_k(\eta)\chi_k(\eta)\rangle = 0. \end{aligned} \tag{7.16}$$

which gives us a coupled equation for f_k and $\langle \chi_k \chi_k \rangle$.
Unfortunately it is very hard to find a general solution to 7.16 for f_k, considering the general form of the BD two point function. Therefore we will take limiting values of the above equation and solve in the IR ($k\eta \ll 1$) and UV ($k\eta \gg 1$), where the BD two point function in the coincidence limit takes the form

$$\langle \chi_k \chi_k \rangle \approx \begin{cases} \frac{1}{k} + \mathcal{O}\left(\eta (k\eta)^{-3}\right) & k\eta \gg 1 \\ \frac{2^{2n}\Gamma(n)^2}{2\pi}\frac{\eta}{(k\eta)^{2n}} + \mathcal{O}\left(\eta (k\eta)^0\right) & k\eta \ll 1 \end{cases}, \tag{7.17}$$

to lowest order in $k\eta$. This again reflects the fact that that the IR regime gives larger contributions than the UV. We have seen this $(k\eta)^{-2n}$ behaviour before, namely in the IR limit of the BD two point function, eq. 6.5.
Let us start with the UV limit. Upon substituting the UV expansion from eq. 7.17 into eq. 7.16 the equation becomes

$$f_k''(\eta) - \frac{2}{\eta} f_k'(\eta) + \frac{1}{\eta^2}\left(k^2\eta^2 - 4\right) f_k(\eta) + \frac{\pi}{3k}\left(4n^2 + 72\xi - 13 - 2k^2\eta^2(1-6\xi)\right) = 0. \quad (7.18)$$

Since $(k\eta)^2 \gg 1$ we can simplify the equation to

$$f_k''(\eta) - \frac{2}{\eta} f_k'(\eta) + k^2 f_k(\eta) - \frac{2\pi}{3}(1-6\xi)k\eta^2 = 0, \quad (7.19)$$

and find the solution

$$\begin{aligned} f_k^{UV}(\eta) =& \frac{C_1\eta^{3/2}}{(k\eta)^{3/2}}\left(k\eta\cos(k\eta) - \sin(k\eta)\right) + \frac{C_2\eta^{3/2}}{(k\eta)^{3/2}}\left(k\eta\sin(k\eta) + \cos(k\eta)\right) \\ &+ \frac{2\pi\eta^3}{3(k\eta)^3}\left(k^2\eta^2 + 2\right)(1-6\xi), \end{aligned} \quad (7.20)$$

where $C_{1/2}$ are complex coefficients. The first two terms of this solution are oscillatory as one would expect in the UV limit. This oscillatory part exactly matches the behaviour of quantised gravitational wave modes and hence describes propagating scalar degrees of freedom [1]. The last term vanishes in the limit of conformal coupling. For fixed (comoving) wavelength modes this solution is fully decaying as time progresses. The dominant contribution will be given by the oscillating terms behaving as

$$f_k^{UV} \sim \frac{C_1\eta^{3/2}}{\sqrt{k\eta}}\cos(k\eta) + \frac{C_2\eta^{3/2}}{\sqrt{k\eta}}\sin(k\eta). \quad (7.21)$$

Hence, we see that here we do not violate out perturbative assumption $sf(x) \ll 1$. If our perturbations are small initially, they will always remain small. Furthermore, there is no simple way to make this solution vanish in the infinite past to get exact de Sitter geometry.
In the IR limit we substitute the IR expansion from eq. 7.17 into eq. 7.16, giving

$$\begin{aligned} &f_k''(\eta) + \frac{2}{\eta} f_k'(\eta) + \frac{1}{\eta^2}\left(k^2\eta^2 - 4\right) f_k(\eta) \\ &\quad - \frac{2^{2n+1}\Gamma(n)^2\eta}{3(k\eta)^{2n}}\left(\frac{k^2\eta^2}{2}(1-6\xi) - 12\xi + (1-12\xi)n^2 + (1-6\xi)n + \frac{9}{4}\right) = 0. \end{aligned} \quad (7.22)$$

Moreover, since $k^2\eta^2 \ll 1$ we can further simplify the equation to

$$\begin{aligned} &f_k''(\eta) + \frac{2}{\eta} f_k'(\eta) - \frac{4}{\eta^2} f_k(\eta) \\ &\quad - \frac{2^{2n+1}\Gamma(n)^2\eta}{3(k\eta)^{2n}}\left(-12\xi + (1-12\xi)n^2 + (1-6\xi)n + \frac{9}{4}\right) = 0. \end{aligned} \quad (7.23)$$

This then leads to the solution

$$f_k^{IR}(\eta) = -\frac{2^{2n}\Gamma(n)^2\left(-12\xi + (1-12\xi)n^2 + (1-6\xi)n + \frac{9}{4}\right)}{3(2-n)(2n+1)}\frac{\eta^3}{(k\eta)^{2n}} + \frac{D_1}{\eta} + D_2\eta^4. \quad (7.24)$$

where $D_{1/2}$ are complex coefficients. By matching our solution to pure de Sitter in the infinite past we can fix the constant $D_2 = 0$. To satisfy this boundary condition we must require

$$f_k(\eta \to \infty) = 0. \tag{7.25}$$

This choice is best justified from cosmological standpoint, as described in sec. 5.1. First we want to have a stage of inflation and then a mechanism to exit this stage, a graceful exit. Since we are in the IR limit but want to take $\eta \to \infty$ we must compensate by taking very low momentum modes $k \ll 1/\eta$.

As we take the comoving momentum $k\eta \to 0$, while fixing η to be finite, we run into a singularity. But as $n \leq 3/2$, the most dominant term is the one proportional to $1/\eta$, which diverges in the asymptotic future, taking $\eta \to 0$. This term violates de Sitter invariance explicitly and will sooner or later also violate the perturbative assumption that $sf(x) \ll 1$. When $sf(x) \sim \mathcal{O}(1)$ the perturbation will become dominant and perturbation theory breaks down. Therefore, in this regime large backreaction could occur. Although this is not a direct conclusion of this calculation, since when perturbation theory breaks down the higher order terms contribute significantly and can change the behaviour of our solution in an unpredictable way.

Chapter 8

Conclusions

The most interesting feature of quantising a quantum field with respect to a dynamical background falls back on the lack of a global timelike Killing vector. This prevents us from diagonalising the Hamiltonian at all times and therefore finding a unique Fock space. We restricted the class of possible vacua by requiring de Sitter invariance, leading first to the well known two parameter class of Mottola-Allen or (α, β)-vacua, invariant under the time preserving part of the de Sitter group. Additionally, we found the one parameter class of α-vacua, where the anticommutator and Feynman Green functions are fully de Sitter invariant. Furthermore, by UV matching onto the Minkowski theory, one fixes the last parameter α and obtains the BD or euclidean vacuum.

There were attempts to set further restrictions or to even exclude some of the (α, β)-vacua [16, 18]. To my awareness, there are no significantly convincing arguments, other than the ones given here, to exclude any of the (α, β)-vacua on physical or mathematical grounds [17]. Therefore, these alternatives can not be ruled out and must be taken seriously.

This freedom is also sensible from a physical point of view. On a given time-slice Σ_1 the de Sitter metric is static, we can diagonalise our Hamiltonian, build a Fock space and do all the beautiful calculations we are used to. On a different time-slice Σ_2 a new Fock space will correspond to the true vacuum. Therefore, our results from Σ_1 will differ from the ones on Σ_2. By allowing for a free parameter α we can transform between two such time-slices. For example, we have seen that for the BD results $\alpha_{BD} = 0$. Hence, for any general time-slice Σ_i we have some α_i, for which the Hamiltonian is diagonalised.

One interesting aspect for further discussion would be to separate the α-vacua in classes connecting the quantisation on different time-slices. Depending on the initial vacuum choice, a different class of parameters should relate the subsequent vacua to each other. If we want to include the BD vacuum, it might be possible to exclude some α parameters.

With the vacuum ambiguity in mind, we began to investigate if quantum effects can in principle significantly backreact and break the de Sitter geometry. The physical motivation came from the idea to provide a new mechanism for a graceful exit due to quantum effects.

What we have found is that as long as we do not break de Sitter isometry the UV regularised contribution to the BD quantum expectation value of the EMT is just a finite constant (see eq. 4.71). Any other α-vacuum choice only gives an additional shift to the mentioned constant value. Depending on the potential or in our case the effective mass of the free field, this contribution can either enhance or suppress the effect of the cosmological constant. The scale factor will still grow exponentially, since we have not broken de Sitter invariance

$$a(t) \propto e^{\sqrt{\frac{1}{3}(\Lambda+8\pi\langle T^{00}\rangle)}\,t}. \tag{8.1}$$

Depending on the sign and magnitude of $\langle T^{00}\rangle$ we screen the cosmological constant and in return influence the speed of expansion of the universe accordingly. When the quantum contribution to the energy density becomes so negative that it dominates over the cosmological constant we obtain an oscillating behaviour of our scale factor and enter the class of *cyclic universes*.
On the other hand, this contribution is constant and might as well be absorbed into the cosmological constant. Thus, as long as de Sitter isometry is respected any contribution to the EMT is constant and de Sitter geometry will not be disturbed. These contributions must not even be negligibly small, as long as it remains smaller than the cosmological constant. In return, we used this as a motivation to find mechanisms which break de Sitter isometry and then obtain contributions to the EMT, which might cause significant backreaction.
By investigating $\lambda\phi^4$ interactions and loop corrections to the Keldysh propagator in eq. 6.64, one finds that de Sitter isometry is still respected. Furthermore, we discovered that as long as we restrict to the expanding Poincaré patch de Sitter isometry will generally be respected for loop corrections to the BD propagator. For any other vacuum choice, the arguments presented in sec. 6.4 break down and de Sitter invariance can be broken, which is also an interesting point for further discussion. Additionally, also in different coordinate patches of the spacetime, de Sitter invariance is broken at loop level. There are ongoing investigations in this direction and the general idea of breaking out of de Sitter spacetime [6, 7, 9, 37, 41–45].
So far we have restricted to exact de Sitter geometry and have not allowed for dynamic backreaction on the spacetime. By allowing for a scalar metric perturbation we broke de Sitter isometry explicitly. Our results give decaying and oscillatory behaviour of the perturbation on short scales in eq. 7.20. One interesting aspect here is the similarity with the gravitational wave modes [1]. On large scales, we find in eq. 7.24 that one part of the solution actually grows, indicating the breakdown of perturbation theory at some point in time. Therefore, our results have to be interpreted with caution. One point that we can nonetheless make is, since in the UV regime the amplitude of the metric perturbation is decaying, to get significant contributions on large scales, the initial amplitude of the perturbation must be large enough to survive this decaying regime. However, if the initial perturbations have to be large, to survive over time and finally grow when stretched to the IR, we also violate our perturbative assumption in the UV limit.
The question of a de Sitter breaking has become a fundamental question of cosmology over the past century. Historically, the first mechanism to do so was discovered by Starobinsky [29]. It is particularly beautiful when discussed in the realms of slow roll inflation and an inflaton field, giving rise to a non-eternal de Sitter solution through a simple scalar degree of freedom. Nonetheless, other mechanisms of de Sitter breaking are also interesting and important. Despite inflation being a widely accepted paradigm, it comes with its own set of unsolved problems. Alternative theories which tell us a completely different story about the history of our universe might grow in popularity in the future. As the de Sitter stage acts as a very efficient classical smoothing mechanism [1], it describes a probably inevitable stage of our universe in any acceptable paradigm. Therefore, the research presented in part here might become very valuable in understanding mechanisms of transition between different stages of the universe we live in.

Appendix A

Conventions

For a general metric $g_{\alpha\beta}$, the inverse is defined via $g_{\alpha\beta}g^{\beta\gamma} = \delta^{\gamma}_{\alpha}$. Throughout this text, the signature $(+-\ldots-)$ will be used, where $\ldots$ reflects that mostly we will keep things general and work in d dimensions.
The conventional Einstein equation is given by

$$G_{\alpha\beta} - \Lambda g_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R - \Lambda g_{\alpha\beta} = 8\pi T_{\alpha\beta}, \tag{A.1}$$

where $R_{\alpha\beta}$ is the Ricci tensor, R the Ricci scalar or scalar curvature, $T_{\alpha\beta}$ the energy momentum tensor (EMT) and Λ the cosmological constant. The Ricci tensor is given by the contraction of the Riemann tensor, and can be expressed as

$$R_{\alpha\beta} = \Gamma^{\gamma}_{\alpha\beta,\gamma} - \Gamma^{\gamma}_{\gamma\alpha,\beta} + \Gamma^{\gamma}_{\gamma\rho}\Gamma^{\rho}_{\alpha\beta} - \Gamma^{\gamma}_{\alpha\rho}\Gamma^{\rho}_{\gamma\beta}, \tag{A.2}$$

where $(\,.\,)_{,\alpha} \equiv \frac{\partial(\,.\,)}{\partial x^{\alpha}}$. The Christoffel symbols $\Gamma^{\alpha}_{\beta\gamma}$ can be written in terms of the metric as

$$\Gamma^{\alpha}_{\beta\gamma} = \frac{1}{2}g^{\alpha\rho}\left(g_{\alpha\rho,\beta} + g_{\beta\rho,\alpha} - g_{\alpha\beta,\rho}\right). \tag{A.3}$$

We generally assume torsion freeness, so that $\Gamma^{\alpha}_{\beta\gamma} = \Gamma^{\alpha}_{\gamma\beta}$. The Ricci scalar is the contraction of the Ricci tensor

$$R = g^{\alpha\beta}R_{\alpha\beta}. \tag{A.4}$$

The EMT obeys the conservation law,

$$\nabla_{\alpha}T^{\alpha\beta} = T^{\alpha\beta}{}_{,\alpha} + \Gamma^{\alpha}_{\alpha\gamma}T^{\gamma\beta} + \Gamma^{\beta}_{\alpha\gamma}T^{\alpha\gamma} = 0, \tag{A.5}$$

where ∇_{α} represents the covariant derivative which in this convention has the property $\nabla_{\alpha}g^{\alpha\beta} = 0$. This conservation law, together with eq. A.1 implies that also the Einstein tensor is conserved, $\nabla_{\alpha}G^{\alpha\beta} = 0$.
Often it will be useful to simplify the notation for 2π-normalised integrals, which we will denote by

$$\int \frac{d^d k}{(2\pi)^{d/2}} \equiv \int đ^d k. \tag{A.6}$$

Additionally, we will usually work in Planck units, setting $\hbar = c = G = 1$.

Appendix B

Squeezed states

We have claimed that the unitary transformation

$$a_{\boldsymbol{k}}^{(\alpha,\beta)} = B^{(\alpha,\beta)}\, a_{\boldsymbol{k}}\, B^{(\alpha,\beta),\dagger} \quad \text{with}$$
$$B^{(\alpha,\beta)} := \exp\left(\frac{1}{2}\int d^{d-1}k\,\alpha\left(e^{i\beta}a_{\boldsymbol{k}}^{\dagger 2} - e^{-i\beta}a_{\boldsymbol{k}}^{2}\right)\right), \tag{B.1}$$

is equivalent to the transformation

$$a_{\boldsymbol{k}}^{(\alpha,\beta)} = \cosh\alpha\, a_{\boldsymbol{k}} - \sinh\alpha\, e^{i\beta}\, a_{\boldsymbol{k}}^{\dagger}. \tag{B.2}$$

Here we want to give the proof of this statement based on [16, 17]. Let us define

$$A := \frac{1}{2}\int d^{d-1}k\,\alpha\left(e^{i\beta}a_{\boldsymbol{k}}^{\dagger 2} - e^{-i\beta}a_{\boldsymbol{k}}^{2}\right), \tag{B.3}$$

so that we can write

$$B^{(\alpha,\beta)}\, a_{\boldsymbol{k}}\, B^{(\alpha,\beta),\dagger} = e^{A}a_{\boldsymbol{k}}e^{-A} = e^{C_A}a_{\boldsymbol{k}} \tag{B.4}$$

where $C_A a_{\boldsymbol{k}} := [C_A, a_{\boldsymbol{k}}]$. One can easily compute

$$[A, a_{\boldsymbol{k}}] = -\alpha e^{i\beta}a_{\boldsymbol{k}}^{\dagger} \quad \text{and} \quad \left[A, a_{\boldsymbol{k}}^{\dagger}\right] = -\alpha e^{-i\beta}a_{\boldsymbol{k}}. \tag{B.5}$$

From which we find

$$C_A^{2n}a_{\boldsymbol{k}} = \alpha^{2n}a_{\boldsymbol{k}} \quad \text{and} \quad C_A^{2n+1}a_{\boldsymbol{k}} = -\alpha^{2n+1}e^{i\beta}a_{\boldsymbol{k}}^{\dagger}. \tag{B.6}$$

Then we obtain from eq. B.4 that

$$\begin{aligned} a_{\boldsymbol{k}}^{(\alpha,\beta)} &= a_{\boldsymbol{k}}\sum_n \frac{\alpha^{2n}}{2n!} - a_{\boldsymbol{k}}^{\dagger}\sum_n e^{i\beta}\frac{\alpha^{2n+1}}{(2n+1)!} \\ &= \cosh\alpha\, a_{\boldsymbol{k}} - \sinh\alpha\, e^{i\beta}\, a_{\boldsymbol{k}}^{\dagger}, \end{aligned} \tag{B.7}$$

which completes the proof.

Appendix C

A review of Green functions in flat space

The action of a free, massive scalar field in d-dimensional Minkowski spacetime is given by

$$S[\phi] = \frac{1}{2}\int d^dx \left(\eta^{\mu\nu}\partial_\mu\phi(x)\partial_\nu\phi(x) - m^2\phi(x)^2\right). \tag{C.1}$$

This action gives rise to the equation of motion

$$\left[\Box_x + m^2\right]\phi(x) = 0, \tag{C.2}$$

where the subscript means that the operator is acting on the coordinate functions x. Then we can write the Green function equation as

$$\left[\Box_x + m^2\right]\mathcal{G}(x,y) = \delta^d(x-y). \tag{C.3}$$

In momentum space we can find the general integral representation of the Green function

$$\mathcal{G}(x,y) = -\int \frac{d^dk}{(2\pi)^d}\frac{e^{-ik.(x-y)}}{k^2-m^2}, \tag{C.4}$$

where $k.x := \eta_{\mu\nu}k^\mu x^\nu$ and $\eta_{\mu\nu}$ is the Minkowski metric. Note that the Fourier transformation is only possible since the Minkowski metric is independent of any coordinate function. We can perform the k^0 integral in the complex plane, deforming the contour around the poles at $k^0 = \pm\sqrt{\boldsymbol{k}^2 + m^2}$. The pole structure of the propagators is important as the physical mass of a particle is determined by the position of the pole $k^2 = m_{ph}^2$ (where the subscript indicates that $m \neq m_{ph}$ necessarily, due to loop corrections) [46, 47].
The choice of contour is ambiguous, giving rise to a variety of Green functions. This choice also reflects the boundary condition imposed on the solution. The possible choices are divided into two classes. The first class provides a solution to the inhomogeneous equation in eq. C.3 [15, 26]:

1. *Retarded Green function*: We choose the contour to run above the poles, as in fig. C.1. For $x^0 - y^0 = t_x - t_y < 0$ one closes the contour in the upper half plane (UHP), resulting in $\mathcal{G}(x,y) = 0$. On the other hand when $t_x - t_y > 0$ one closes in the lower half plane (LHP). Now both poles are enclosed within the contour leading to a non-zero result, namely the *retarded Green function* $G_R(x,y)$. Hence, for fixed y, $G_R(x,y)$ has support for x in the future light cone of y.

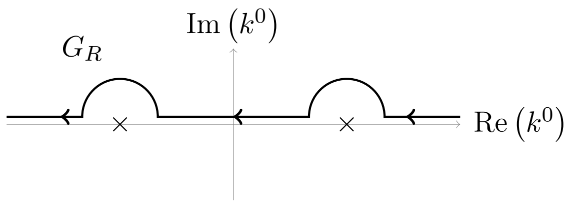


Figure C.1: The contour of the retarded Green function G_R.

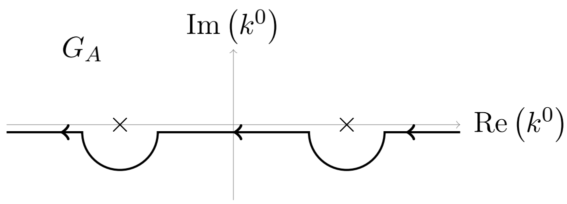


Figure C.2: The contour of the advanced Green function G_A.

2. *Advanced Green function*: This is the complement of the retarded Green function. Here the contour is chosen to run below the poles, as in fig. C.2. Here, only when $t_x - t_y < 0$ both poles are enclosed, giving the *advanced Green function* $G_A(x, y)$. For fixed y, $G_A(x, y)$ has support for x in the past light cone of y.

3. *Averaged Green function*: The average of the retarded and advanced Green functions is denoted
$$\bar{G}(x, y) = \frac{1}{2}\left[G_R(x, y) + G_A(x, y)\right], \tag{C.5}$$
and is given by the contour in fig. C.3. Due to the support properties of the retarded and advanced Green functions, the support of $\bar{G}(x, y)$ is given for x in the full light cone of y.

4. *Feynman Green function*: The Feynman Green function $G_F(x, y)$ is obtained by choosing the contour displayed in fig. C.4, namely going under the left pole but above the right one.

The second type of contour choice is given by differences of the above Green functions, which consequently provide solutions to the homogeneous equation
$$\left[\Box_x + m^2\right] \mathcal{G}(x, y) = 0. \tag{C.6}$$
Due to the linearity of the differential operator, the δ-functions on the right hand side simply cancel. In this class we have [15, 26]:

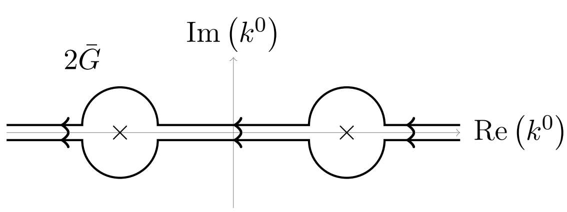


Figure C.3: The contour of the averaged Green function $\bar{G}$.

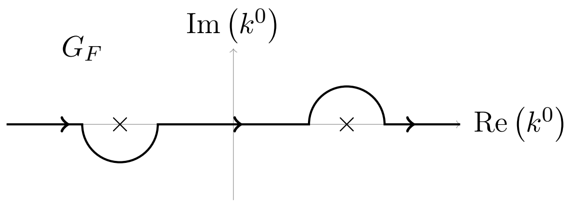


Figure C.4: The contour of the retarded Green function G_R.

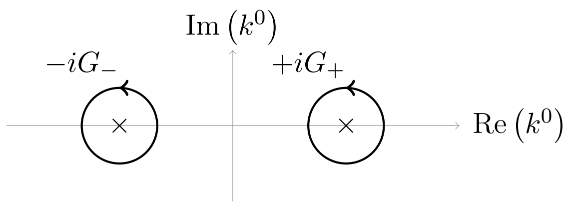


Figure C.5: The contour of the Wightman function G_+ and its conjugate G_-.

5. *Wightman function and conjugate*: The Wightman function, $G_+(x,y)$, is obtained by circling the right pole, which contributes $+iG_+(x,y)$. It is the difference of advanced and Feynman Green function (actually the sum in this case, but note the different contour directions),

$$+iG_+(x,y) = -G_A(x,y) - G_F(x,y). \tag{C.7}$$

The left pole contributes $-iG_-(x,y)$, where $G_-(x,y) = G_+(x,y)^*$ and

$$-iG_-(x,y) = -G_R(x,y) + G_F(x,y). \tag{C.8}$$

The contours are shown in fig C.5.

6. *Commutator Green function*: The commutator Green function, $G(x,y)$, is obtained by circling both poles, as in fig. C.6. It is given by the difference

$$\begin{aligned} G(x,y) &= G_A(x,y) - G_R(x,y) \\ &= -i\left[G_+(x,y) - G_-(x,y)\right] \\ &= 2\text{Im}\left(G_+(x,y)\right) \end{aligned} \tag{C.9}$$

7. *Anticommutator Green function*: The anticommutator Green function $G_+(x,y)$ is ob-

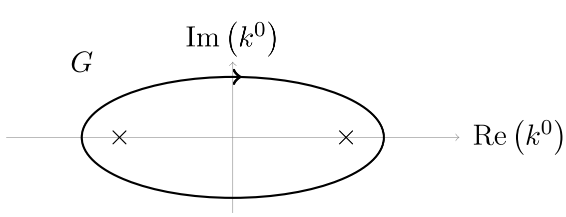


Figure C.6: The contour of the commutator Green function G.

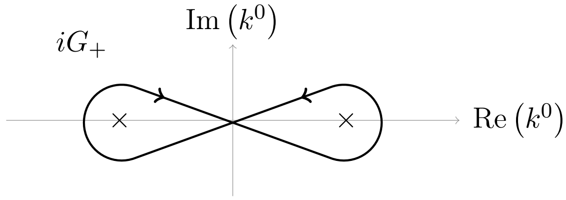


Figure C.7: The contour of the anticommutator Green function G_+.

tained via the contour displayed in fig C.7. It is given by

$$\begin{aligned} iG(x,y) &= G_+(x,y) + G_-(x,y) \\ &= 2\mathrm{Re}\left(G_+(x,y)\right) \\ &= 2i\left[\bar{G}(x,y) + G_F(x,y)\right] \end{aligned} \tag{C.10}$$

The above Green functions have another interpretation. They have a direct connection to the two point functions (see eg. [23]),

$$\begin{aligned} G_+(x,y) &= \langle vac \,|\, \phi(x)\phi(y) \,|\, vac\rangle \\ G_-(x,y) &= \langle vac \,|\, \phi(y)\phi(x) \,|\, vac\rangle \,, \end{aligned} \tag{C.11}$$

for some vacuum $\big| vac\big\rangle$. Hence, the commutator and anticommutator Green functions can be written as

$$\begin{aligned} iG(x,y) &= \langle vac \,|\, [\phi(x), \phi(y)] \,|\, vac\rangle \\ G^{(1)}(x,y) &= \langle vac \,|\, \{\phi(x), \phi(y)\} \,|\, vac\rangle \end{aligned} \tag{C.12}$$

For the Feynman Green function,

$$G_F(x,y) = -\bar{G}(x,y) - \frac{1}{2} i G^{(1)}(x,y). \tag{C.13}$$

We can therefore write it as

$$\begin{aligned} iG_F(x,y) &= \theta(t_x - t_y) G_+(x,y) + \theta(t_y - t_x) G_-(x,y) \\ &= \langle vac \,|\, \mathcal{T}\{\phi(x)\phi(y)\} \,|\, vac\rangle \,, \end{aligned} \tag{C.14}$$

where $\mathcal{T}\{.\}$ represents the time ordered product.

Appendix D

The two point function by direct calculation

In this section we will present the direct calculation of obtaining an analytic expression for the two point function in the BD vacuum, given by the mode expansion, in eq. 3.39.
We start by substituting the mode expansion 3.39 into the expression for the two-point function,

$$\begin{aligned} G_+(x,y) &= \langle 0 \,|\, \phi(x)\phi(y) \,|\, 0\rangle \\ &= \frac{\pi}{4H}(H^2\eta_x\eta_y)^{\frac{d-1}{2}} \int \frac{d^{d-1}k}{(2\pi)^{d-1}} H_n^{(1)}(k\eta_x)^* H_n^{(1)}(k\eta_y) e^{-i\boldsymbol{k}.(\boldsymbol{x}-\boldsymbol{y})}. \end{aligned} \tag{D.1}$$

It is practical to now go to spherical coordinates, using $d^{d-1}k = k^{d-2}\sin^{d-3}\theta\, dk\, d\theta\, d\Omega_{d-3}$ and orienting our coordinate system such that $\boldsymbol{k}.(\boldsymbol{x}-\boldsymbol{y}) = kr\cos\theta$, where $r = |\boldsymbol{x}-\boldsymbol{y}|$. We can then write the angular integral in terms of a Bessel function,

$$\begin{aligned} \int_0^\pi d\theta \sin^{d-3}\theta\; e^{-ikrcos\theta} &= \int_{-1}^1 ds(1-s^2)^{\frac{d-4}{2}}\; e^{-ikrx} \\ &= \sqrt{\pi}\Gamma\left(\frac{d-2}{2}\right)\left(\frac{2}{kr}\right)^{\frac{d-3}{2}} J_{\frac{d-3}{2}}(kr). \end{aligned} \tag{D.2}$$

Substituting this back into our original expression gives,

$$\begin{aligned} G_+^{BD}(x,y) =& \frac{\pi}{4H}(H^2\eta_x\eta_y)^{\frac{d-1}{2}} \frac{\sqrt{\pi}}{(2\pi)^{d-1}}\Omega_{d-3}\Gamma\left(\frac{d-2}{2}\right)\left(\frac{2}{r}\right)^{\frac{d-3}{2}} \\ &\times \int_0^\infty dk H_n^{(2)}(k\eta_x) H_n^{(1)}(k\eta_y) J_{\frac{d-3}{2}}(kr) k^{\frac{d-1}{2}}, \end{aligned} \tag{D.3}$$

where $\Omega_{d+1} = \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}$. This integral can be solved analytically as follows [48],

$$\begin{aligned} &\int_0^\infty dk\, H_n^{(2)}(k\eta_x) H_n^{(1)}(k\eta_y) J_{\frac{d-3}{2}}(kr) k^{\frac{d-1}{2}} \\ &= \frac{4}{\pi^2}\int_0^\infty dk\, K_n(ik\eta_x(1-i\epsilon\,\mathrm{sgn}(x,y))) K_n(-ik\eta_y(1-i\epsilon\,\mathrm{sgn}(x,y))) J_{\frac{d-3}{2}}(kr) k^{\frac{d-1}{2}} \\ &= \frac{4\sqrt{\pi}}{2^{3/2}\pi^2} \frac{\Gamma\left(\frac{d-1}{2}+n\right)\Gamma\left(\frac{d-1}{2}-n\right)}{(\eta_x\eta_y)^{\frac{d-1}{2}}\left(Z_\epsilon^2-1\right)^{\frac{d-2}{4}}} r^{\frac{d-3}{2}} P_{n-\frac{1}{2}}^{-\frac{d-2}{2}}(-Z_\epsilon), \end{aligned} \tag{D.4}$$

where $P_l^m(x)$ is the associated Legendre function of the first kind and

$$\operatorname{sgn}(x,y) \coloneqq \Theta(\eta_x - \eta_y) - \Theta(\eta_y - \eta_x). \tag{D.5}$$

It is required to add a small imaginary part $-i\epsilon \operatorname{sgn}(x,y)$ to η_x and η_y such that

$$\begin{aligned} &\mathrm{Re}\,(i\eta_x(1 - i\epsilon \operatorname{sgn}(x,y)) + (-i)\eta_y(1 - i\epsilon \operatorname{sgn}(x,y))) = \\ &\qquad = (\eta_x - \eta_y)\epsilon \operatorname{sgn}(x,y) > |\mathrm{Im}\,(r)| = 0, \end{aligned} \tag{D.6}$$

which is a requirement for the solution in D.4 and defines the $i\epsilon$-prescription here at the same time [7, 48]. Hence, in Poincaré coordinates and including the $i\epsilon$ prescription

$$Z_\epsilon(x,y) \coloneqq 1 + \frac{(\eta_x - \eta_y)^2 - r^2}{2\eta_x\eta_y} - i\epsilon \operatorname{sgn}(x,y). \tag{D.7}$$

Putting everything together, we find the BD two point function

$$G_+(Z_\epsilon) = \frac{H^{d-2}}{2(2\pi)^{d/2}} \frac{\Gamma\left(\frac{d-1}{2} + n\right)\Gamma\left(\frac{d-1}{2} - n\right)}{(Z_\epsilon^2 - 1)^{\frac{d-2}{4}}} P^{-\frac{d-2}{2}}_{n-\frac{1}{2}}(-Z_\epsilon). \tag{D.8}$$

We can write this result in terms of the hypergeometric function using [39, Eq. 14.3.15]

$$P^{-\frac{d-2}{2}}_{n-\frac{1}{2}}(-Z_\epsilon) = \frac{2^{-\frac{d}{2}+1}(Z_\epsilon^2 - 1)^{\frac{d-2}{4}}}{\Gamma\left(\frac{d}{2}\right)} {}_2F_1\left(\frac{d-1}{2} - n, \frac{d-1}{2} + n; \frac{d}{2}; \frac{1+Z_\epsilon}{2}\right). \tag{D.9}$$

Therefore, our final expression for the BD two point function is

$$G_+(Z_\epsilon) = \frac{H^{d-2}}{(4\pi)^{d/2}} \frac{\Gamma(N_-)\Gamma(N_-)}{\Gamma\left(\frac{d}{2}\right)} {}_2F_1\left(N_-, N_+; \frac{d}{2}; \frac{1+Z_\epsilon}{2}\right), \tag{D.10}$$

where $N_\pm \coloneqq \frac{d-1}{2} \pm n$. This is shown in fig. 3.1 in $d = 4$ for different values of m^2/H^2.

Appendix E

Time ordering in the path integral

Here, we want to show that time ordering appears naturally in the path integral formalism. This can be seen by considering the two point correlation function

$$\begin{aligned}&\int_{\phi(x_i)}^{\phi(x_f)} \mathcal{D}\phi\ \phi(x_1)\phi(x_2)e^{iS[\phi]} = \\&= \int d\phi(x_m) \int_{\phi(x_m)}^{\phi(x_f)} \mathcal{D}\phi\ e^{iS_f[\phi]} \int_{\phi(x_i)}^{\phi(x_m)} \mathcal{D}\phi\ e^{iS_i[\phi]}\phi(x_1)\phi(x_2),\end{aligned} \tag{E.1}$$

where we have split the path integral at a certain point in time such that $t_i < t_1 < t_m < t_2 < t_f$, but one could equally well interchange the ordering of t_1 and t_2. The main point is that t_m lies between t_1 and t_2. When splitting the path integral, one must integrate separately over the intermediate configuration $\phi(x_m)$. Also when have split the integration of the action so that

$$S_i = \int_{x_i}^{x_m} d^dx\mathcal{L} \quad \text{and} \quad S_f = \int_{x_m}^{x_f} d^dx\mathcal{L}, \tag{E.2}$$

where $\mathcal{L}$ is the Lagrangian of the theory. Now since in our example $t_m < t_2$ we can rearrange the above integral as

$$\begin{aligned}&\int_{\phi(x_i)}^{\phi(x_f)} \mathcal{D}\phi\ \phi(x_1)\phi(x_2)e^{iS[\phi]} = \\&= \int d\phi(x_m) \left(\int_{\phi(x_m)}^{\phi(x_f)} \mathcal{D}\phi\ e^{iS_f[\phi]}\phi(x_2) \right) \left(\int_{\phi(x_i)}^{\phi(x_m)} \mathcal{D}\phi\ e^{iS_i[\phi]}\phi(x_1) \right),\end{aligned} \tag{E.3}$$

where we were free to move the field operator $\phi(x_2)$ through the first integral, as the expressions in the brackets correspond to complex numbers only. putting everything back together, one obtains

$$\frac{\int_{\phi(x_i)}^{\phi(x_f)} \mathcal{D}\phi\ \phi(x_1)\phi(x_2)e^{iS[\phi]}}{\int_{\phi(x_i)}^{\phi(x_f)} \mathcal{D}\phi\ e^{iS[\phi]}} = \frac{\langle out \,|\, \phi(x_2)\phi(x_1) \,|\, in\rangle}{\langle out \,|\, in\rangle}, \tag{E.4}$$

where the denominator on the left hand side in necessary for correct normalisation. If we would have taken the ordering $t_i < t_2 < t_m < t_1 < t_f$, we would obtain the same expression with $\phi(x_1)$ and $\phi(x_2)$ exchanged. Hence, in general

$$\begin{aligned}\frac{\langle out\,|\,\mathcal{T}\{\phi(x_1)\phi(x_2)\}\,|\,in\rangle}{\langle out\,|\,in\rangle} &= \frac{\int_{\phi(x_i)}^{\phi(x_f)}\mathcal{D}\phi\ \phi(x_1)\phi(x_2)e^{iS[\phi]}}{\int_{\phi(x_i)}^{\phi(x_f)}\mathcal{D}\phi\ e^{iS[\phi]}} \\ &= \frac{1}{\sqrt{|g(x_1)|}}\frac{\delta}{\delta iJ(x_1)}\frac{1}{\sqrt{|g(x_2)|}}\frac{\delta}{\delta iJ(x_2)}\ln Z[J,g_{\mu\nu}]\bigg|_{J=0},\end{aligned} \tag{E.5}$$

we obtain a natural time ordering. Of course one can easily generalise this to any number of fields.

Appendix F

Tensors respecting de Sitter isometry

Following Weinberg [10] we can show that if we demand to fully respect de Sitter isometry, any rank-two tensor is proportional to the metric tensor.
A tensor $A_{\mu\nu\dots}(x)$ is said to be *maximally form invariant* if the Lie derivative with respect to any of the Killing vectors of the spacetime $\xi^\mu(x)$ vanishes

$$0 = \mathcal{L}_\xi A_{\mu\nu\dots}(x) = A_{\mu\nu\dots,\rho}(x)\xi^\rho(x) + A_{\rho\nu\dots}(x)\xi^\rho{}_{,\mu}(x) + A_{\mu\rho\dots}(x)\xi^\rho{}_{,\nu}(x) + \dots, \tag{F.1}$$

where $_{,\mu} \coloneqq \frac{\partial}{\partial x^\mu}$. That is, under an infinitesimal coordinate transformation

$$x^\mu \to y^\mu = x^\mu - \epsilon\xi^\mu(x) \tag{F.2}$$

the considered tensor remains invariant

$$A^{'}_{\mu\nu\dots}(x) - A_{\mu\nu\dots}(x) = \mathcal{L}_{\epsilon\xi} A_{\mu\nu\dots}(x) = 0. \tag{F.3}$$

We now choose a Killing vector which at a given point X satisfies

$$\xi^\mu(X) = 0 \quad \text{and} \quad \xi_{\rho,\sigma} = g_{\rho\lambda}(X)\xi^\lambda{}_{,\sigma}(X) \tag{F.4}$$

forms an antisymmetric tensor. Then the above condition reads

$$0 = \mathcal{L}_{\epsilon\xi} A_{\mu\nu\dots}(x) = \xi_{\lambda,\sigma}(X)\left(\delta^\sigma_\mu A^\lambda{}_{\nu\dots}(X) + \delta^\sigma_\nu A_\mu{}^\lambda{}_{\dots}(X) + \dots\right) \tag{F.5}$$

where the terms in the bracket must be symmetric in $\sigma \leftrightarrow \lambda$ as $\xi_{\lambda,\sigma}$ is antisymmetric per definition. Since X is arbitrary, one can form this argument at any point and hence it is valid everywhere.
For a rank-two tensor the above condition simplifies to

$$\delta^\sigma_\mu B^\lambda{}_\nu + \delta^\sigma_\nu B_\mu{}^\lambda = \delta^\lambda_\mu B^\sigma{}_\nu + \delta^\lambda_\nu B_\mu{}^\sigma. \tag{F.6}$$

Contracting σ with μ and lowering λ gives

$$(d-1)B_{\lambda\nu} + B_{\nu\lambda} = g_{\lambda\nu} B_\mu{}^\mu. \tag{F.7}$$

Now subtracting the same expression with exchanged indices

$$(d-2)\left(B_{\lambda\nu} - B_{\nu\lambda}\right) = 0. \tag{F.8}$$

Hence for $d \neq 2$, $B_{\lambda\nu}$ is symmetric and hence from eq. F.7

$$B_{\lambda\nu} = g_{\lambda\nu}\frac{B_\mu{}^\mu}{d} =: g_{\lambda\nu}C. \tag{F.9}$$

The last check one has to preform is, to show that C is coordinate independent. To achieve this, we demand that the Lie derivative condition holds for our obtained result

$$0 = \mathcal{L}_{\epsilon\xi}(g_{\lambda\nu}C) = g_{\lambda\nu}\mathcal{L}_{\epsilon\xi}C = g_{\lambda\nu}\epsilon\xi^\mu C_{,\mu}, \tag{F.10}$$

since the Lie derivative of the metric tensor is zero. Since we are free to choose the value of our Killing vector at any point in a maximally symmetric space,

$$C_{,\mu} = 0, \tag{F.11}$$

and we have shown coordinate independence of C. Therefore, in maximally symmetric spaces the only rank-two tensor which respects the full isometry of the space is the metric tensor times a coordinate independent quantity.

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Erklärung

Hiermit erkläre ich, die vorliegende Arbeit selbständig verfasst zu haben und keine anderen als die in der Arbeit angegebenen Quellen und Hilfsmittel benutzt zu haben.

München, September 11, 2020

Unterschrift